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FOUNDATIONS, THEORY OF SETS, LOGIC

Hintikka, K. Jaakko J. Identity, variables, and impredicative definitions. J. Symb. Logic 21 (1956), 225–245.

The starting point of this investigation is the interpretation of the reference to "two" objects in either ordinary or formal languages. Is the phrase in question supposed to include the possibility of the "two" objects being identical (inclusive interpretation) or must they be distinct (exclusive interpretation)? More formally, does the assertion of the existence of a certain object as expressed by an existential quantifier include the possibility that this object is referred to by one of the preceding quantifiers? The standard version of the lower predicate calculus implies that it does, whereas in the present paper the author introduces two further pairs of quantifiers which are, to a greater or lesser extent, "exclusive" in the sense indicated above. Rules of transformation and deduction are given for the resulting calculi and it is stated without proof that the systems thus obtained are complete. Finally, it is shown that the corresponding reinterpretation of the principle of abstraction in set theory eliminates Russell's paradox and does not seem to lead to any other contradictions. *A. Robinson.*

Quine, W. V. On Frege's way out. Mind 64 (1955), 145–159.

In attempting to avoid Russell's paradox, Frege abandoned the abstraction principle (A):

$$(y)[y \in \hat{x}(\varphi x) . \equiv \varphi y],$$

and replaced it by the more restrictive principle (B): $(y)[y \in \hat{x}(\varphi x) . \equiv . y \neq \hat{x}(\varphi x) . \varphi y]$. Quine shows that a contradiction can be derived from $(\exists x)(\exists y)(x \neq y)$ and the following weaker form of (B):

$$(C): \quad (y)[y \neq \hat{x}(\varphi x) . \supset : y \in \hat{x}(\varphi x) . \equiv \varphi y].$$

Thus, Frege's "way out" is no way out at all. [For a discussion of variants of Frege's proposal see the paper reviewed above.]

The author also briefly discusses various forms of axiomatic set theory in terms of the way in which each system modifies the (inconsistent) abstraction principle: $(\exists y)(x)(x \in y \equiv \varphi x)$. In an appendix, it is demonstrated that the following extensionality law for attributes: $(x)(\varphi x \equiv \psi x) \supset \varphi = \psi$, which is rejected in the Principia, is a consequence of the (possible) rule of inference: If $\vdash . \varphi x \equiv \psi x$, then $\vdash . \varphi = \psi$. This is taken to be an argument for avoidance of the notion of attributes.

E. Mendelson (Cambridge, Mass.).

Kol'man, È. On the philosophical significance of the geometry of Lobačevskiĭ. Vestnik Akad. Nauk SSSR 26 (1956), no. 6, 72–77. (Russian)

The essential content of this note is that the concepts arising from sensual evidence are not sufficient for mathematical (more generally: scientific) development. They must be transcended: there must be added the abstracting thought of man. This principle, which is traced to John Locke (1632–1704), assures the opposition of scientific productiveness to any conservatism and to all stagnation of thought. "The genuine philosophic meaning of the geometry of Lobačevskiĭ consists in the assertion of the power of the abstracting thought of man, with the help of which he overcomes not only space, time and inert matter, but also the restriction of his own nature".

B. Germansky (Jerusalem).

★ **Nitto, Sotaro. The fundamental study of mathematics.** Maruzen Co., Ltd., Tokyo, Japan, 1956. 83 pp. $1.50.

To elucidate the standpoint of the author, it is sufficient to give one characteristic instance. He "concludes" from the associative law of addition that the addition of the infinite series $1-1+1-1+\cdots$ yields

$$1-(1-1)-(1-1)-\cdots=(1-1)+(1-1)+\cdots$$

or $1=0$. Since the last equality is not correct, he infers (citing the words of the author): "$\cdots$ the associative law of addition has come to be cast doubt upon the worth as a universal and fundamental axiom, for its limits of application are narrowed down. A hopeless puzzle are similarly left as to the contradiction between the sum of conditionally convergent series and the universality of the commutative law of addition." It cannot be the task of a reviewer to shell out from such an unscientific and even unlogical bulk of considerations what may be positive in the content of the book. *B. Germansky* (Jerusalem).

Adyan, S. I. Algorithmic unsolvability of problems of recognition of certain properties of groups. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 533–535. (Russian)

Algorithmic unsolvability is established for the recognition of certain properties of groups, in particular: the property of being the unit group, of being a simple group, of having a free subgroup, of being decomposable into the free product of n subgroups for a fixed $n>1$. It is also impossible to find an algorithm that will settle the question for an arbitrary given group whether the word problem is solvable or, to use the expression of the author, "the metaproblem of identity" is unsolvable.

The article also establishes the algorithmic unsolvability of the problem of isomorphism in the following strong formulation: let F_0 be a given arbitrary group. It is impossible to find an algorithm that will decide for every group F whether it is isomorphic to the group F_0 or not.

For the formulation of the remaining results the following concepts are essential: a property of a group is called invariant if every group that is isomorphic to a group possessing the property also possesses the property. An invariant property is called hereditary if every subgroup of a group possessing the property also possesses the property (oversight of the author: in his formulation the

word "invariant" is missing). A hereditary property is called non-trivial if there exists some free group with a finite number of generators that does not possess the property.

The article also establishes the impossibility of an algorithm for deciding whether an arbitrary group F possesses the property $\alpha_0\&\beta_0$, where α_0 is a non-trivial hereditary property and β_0 is an invariant property such that there exists the group F_1 with the property $\alpha_0\&\beta_0$.

It is also impossible to find an algorithm to decide the question whether an arbitrary group possesses a given invariant property α_0 that is possessed by at least one group but is not possessed by any group with unsolvable problem of identity.

It is the purpose of the author to reduce the proof of these assertions to the proof of a single theorem: this theorem is not included in the article under review (the reviewer is acquainted with it only in the manuscript of the author's dissertation). The formulation of this theorem makes use of the group constructed by P. S. Novikov with unsolvable problem of identity (Dokl. Akad. Nauk SSSR **103** (1955), No. 4, 533–535).

A. S. Esenin-Vol'pin (RŽMat **1956**, no. 962).

Pillai, K. C. S. Some results useful in multivariate analysis. Ann. Math. Statist. **27** (1956), 1106–1114.

Consider the function

$$f(x_1, x_2, \cdots, x_k)=\prod\{x_i^q(1-x_i)^r \exp(tx_i)\} \prod_{i<j} (x_i-x_j),$$

where $0\leqq x_1\leqq x_2\leqq\cdots\leqq x<1$ with $q, r>-1$ and t independent of the x's. Now $\prod_{i>j}(x_i-x_j)$ can be expressed as a Vandermonde determinant. Then the ith row of the determinant is multiplied by

$$x_{k-i+1}^q(1-x_{k-i+1})^r \exp(tx_{k-i+1}) \ (i=1, 2, \cdots, k)$$

and integration is indicated for each element with appropriate limits to obtain the integral of $f(x_1, x_2, \cdots, x_k)$. The result is called a pseudo-determinant since the order of integration must be preserved in the expansion.

Integral functions of certain types occur in the expansion. Properties of the pseudo-determinants and of these integral functions are derived or stated in lemmas and theorems. Many of the results given are in the form of recursion formulas for reducing special types of pseudo-determinants in terms of those of lower order. Application is made to certain distribution problems in multivariate analysis — e.g., those relating to the characteristic roots of a determinantal equation. The results are applicable to such problems as the moments of the sum of the roots and the distribution of the extreme roots. *P. S. Dwyer.*

Bagemihl, F. The Baire category of independent sets. Compositio Math. **13** (1956), 71–75.

Associate with every point x of the linear continuum C a subset $P(x)$ of C whose closure does not contain x; a subset of C is called independent if $x\notin P(y)$ and $y\notin P(x)$ for every two of its points x and y. Typical results: There always exists an independent set of second category; hence there always exists an independent set that is of second category in every subinterval of some interval of C. However, there does not always exist an independent set that is everywhere dense in C; hence there does not always exist a residual independent set. In particular, if M is of second category, and $P(x)$ is of second category for every $x\in M$, then no residual independent set exists. Even if every $P(x)$ contains at most one point, there need not exist an independent set that is residual in some interval; and if in addition, the distance from x to $P(x)$ is bounded away from zero, there still need not exist an independent set that is residual in C. The author points out that some of his results are related to results of Fodor about the Lebesgue measure of independent sets [Composition Math. **8** (1951), 250; Nieuw Arch. Wisk. (2) **23** (1951), 247–248; MR **12**, 683; **13**, 525]. *L. Gillman.*

Schwarz, Gideon. A note on transfinite iteration. J. Symb. Logic **21** (1956), 265–266.

For an operator D satisfying (1) $D(x)\supset X$, the αth power (iteration) of D is defined naturally by the formula $D^\alpha(X)=D(\mathbf{U}_{\eta<\alpha} D^\eta(X))$. The following theorem states a sufficient condition that the powers of D be repetitive from some point on. Let μ be a nonlimit ordinal. Let D satisfy (1), (2) $X\supset Y$ implies $D(X)\supset D(Y)$, and (3) for every set A of power $\aleph_\mu$ there exists a set Q of subsets of A of powers $<\aleph_\mu$, such that $D(\mathbf{U}_{a\in A} a)=\mathbf{U}_{B\in Q} D(\mathbf{U}_{a\in B} a)$. Then $D^\alpha(X)=\mathbf{U}_{\eta<\omega_\mu} D^\eta(X)$ for every $\alpha\geqq\omega_\mu$. {Reviewer's remark: the proof works for inaccessible $\aleph_\mu$ as well, i.e., for all regular $\aleph_\mu$.} *L. Gillman* (Lafayette, Ind.).

Fraïssé, Roland. Etude de certains opérateurs dans les classes de relations, définis à partir d'isomorphismes restreints. Z. Math. Logik Grundlagen Math. **2** (1956), 59–75.

Démonstrations et mise au point de résultats contenus dans C. R. Acad. Sci. Paris **240** (1955), 2109–2110, 2191–2193 [MR **16**, 1005, 1006]. *Đ. Kurepa* (Zagreb).

Pickert, Günter. Der Mengen- und Funktionsbegriff in der Anfängervorlesung. Math.-Phys. Semesterber. **5** (1956), 71–79.

For use in introductory courses, the notion of set is carefully introduced as propositional form and then correspondingly the concept of function is discussed.

A. Rosenthal (Lafayette, Ind.).

Suetuna, Zyoiti. Über den Begriff der Totalität in der Mathematik. Ann. Japan Assoc. Philos. Sci. **1** (1956), 33–40.

Ceĭtin, G. S. On the problem of recognition of properties of associative calculuses. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 209–212. (Russian)

This is a continuation of Markov's book [Trudy Mat. Inst. Steklov. **42** (1954); MR **17**, 1038]. Let I be a property of associative calculuses. The recognition problem for I is that of finding a uniform constructive process for testing whether a given calculus has the property I or not. This uniform process is taken to be a fixed normal algorithm which is always applicable to a word describing (in a certain technical sense) any such calculus A; when so applied to the description of A, the algorithm terminates in the null word if A has the property I, and in a non-null word otherwise. Let us call I a $J_{\mathfrak{p}}$-property just when a) I is invariant of isomorphism, b) there exists a calculus A which is not isomorphic to a part of any calculus having the property I, and c) there exists a calculus B in an alphabet with $\mathfrak{p}$ letters which has the property I. The author proves two theorems: 1) If I is a $J_{\mathfrak{p}}$-property its recognition problem is unsolvable in any alphabet having at least $\mathfrak{p}+2$ letters; 2) there exists a $J_{\mathfrak{p}}$-property whose recognition problem is solvable in any alphabet with not more than $\mathfrak{p}+1$ letters. This sharpens somewhat a theorem at the close of Markov's book. *H. B. Curry.*

Mučnik, A. A. On the unsolvability of the problem of reducibility in the theory of algorithms. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 194–197. (Russian)

The results in this paper refer to questions concerning recursively solvable sets that were raised by Post [Bull. Amer. Math. Soc. **50** (1944), 284–366; MR **6**, 29]. One of these questions in particular receives an answer in the first theorem which states that there are two nonrecursive sets which are not reducible to one another by means of a partially recursive operator. This result and its corollaries are obtained from a combination of the theory of functional representations and the calculus of mass problems, both due to Yu. Medvedev [Dissertation, Moskov. Gos. Univ., 1955]. [For a formulation of the calculus of mass problems see Medvedev, Dokl. Akad. Nauk. SSSR (N.S.) **104** (1955), 501–504; MR **17**, 448.]

The author considers at the conclusion of the paper a generalization of reducibility and solvability problems achieved by combining the theory of mass problems and questions of recursive solvability of sets. *E. J. Cogan.*

Trahtenbrot, B. A. Tabular representation of recursive operators. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 417–420. (Russian).

This paper is concerned with numerical functions, i.e. functions from natural numbers, or finite sequences of natural numbers, to natural numbers; and with operators (or functionals) which determine a numerical function ϕ, called the image function, from certain functions $f_1, \cdots, f_n$ called the given functions. Such an operator is called partial recursive if it is defined by a set of equations (or scheme) $\mathfrak{A}$ in which, besides the given and image functions, there may appear certain auxiliary functions $\sigma_1, \cdots, \sigma_v$, such that, given $x_1, \cdots, x_m$ at most one value of $\phi(x_1, \cdots, x_m)$ can be deduced from $\mathfrak{A}$ and the values of the given functions by the rules usual in the theory of recursive functions. The operator is called general recursive if the image function is always defined whenever the given functions are always defined; it is called primitive recursive if ϕ is obtained from $f_1, \cdots, f_n$, and functions $g_k^i(x_1, \cdots, x_k)=x_i$, $g(x_1, \cdots, x_k)=0$ by substitions and primitive recursions; it is called of Post type if it is obtained by specializing some of the given functions of a primitive recursive operation to be particular general recursive functions; and it is called a π-operator if it transforms predicates (functions whose values are 0 or 1 only) into predicates. Given an operator T applied to particular given functions $f_1^0, \cdots, f_n^0$ according to $\mathfrak{A}$, a function $\psi(x_1, \cdots, x_n)$ is called a signalizer of T just when for all $x_1, \cdots, x_m$ there is a deduction of the value of $\phi(x_1, \cdots, x_m)$ in which more of the numbers used exceed $\psi(x_1, \cdots, x_m)$. The author then states without proof theorems which say approximately the following: 1) the class of primitive recursive operators is a proper part of the class of Post operators, and the latter again of the class of recursive operators; 2) if M is a class of functions uniformly bounded by a primitive (general) recursive functions, then every primitive recursive (Post) operator T has a primitive (general) recursive signalizer when applied to functions in M; 3) a π-operator has a truth table representation, in a sense similar to that of E. L. Post [Bull. Amer. Math. Soc. **50** (1944), 284–316; MR **6**, 29], defined by a primitive (general) recursive enumeration of the Gödel numbers of the truth-tables, if and only if it is a primitive recursive (Post) operator; 4) certain generalizations of the result in 3) hold for operators which are not π-operators and even for arbitrary partial recursive operators; 5) a partial recursive operator T which is completely applicable to any recursive f majorized by a given recursive function $g(x)$ is equivalent to a Post operator for such an f; this applies in particular to completely applicable π-operators. *H. B. Curry.*

Rohleder, Hans. Zur Umformung logischer Ausdrücke mit Hilfe programmgesteuerter Rechenanlagen. Z. Math. Logik Grundlagen Math. **2** (1956), 57–58.

In this short note the author considers the possibility of using centrally programmed computers to find minimal normal forms in the predicate calculus. He describes generally his ideas but does not in this paper carry through the details. *H. H. Goldstine* (Princeton, N.J.).

Kokoszyńska, Maria; Kubiński, Tadeusz; and Słupecki, Jerzy. The application of logistic concepts to the explication of some concepts in natural science. Studia Logica **4** (1956), 155–211. (Polish. Russian and English summaries)

An analysis of a few biological notions by means of logical apparatus is presented. The classification of individuals into blood groups is treated by a definition by abstraction. Rigorous definitions are given for the notion of phenotype and a few variants of the notion of genotype. An axiomatisation of the notion of offspring is provided as a bases for the concept of genotype. Genotype is also related to the concept of being equally probable which, though not explained, seems to be connected with strict implication. The paper contains an attack on alleged abuses of logical methods in natural sciences, and on neopositivism in particular. *H. Hiż.*

See also: Ewald, p. 502; Hano and Ozeki, p. 507; Davidson and Suppes, p. 518; Moore and Shannon, p. 549.

ALGEBRA

★**Rey Pastor, J. Lecciones de algebra. [Textbook of algebra.] Fourth edition.** Madrid, 1957. xiii+336 pp. 175 ptas.

This is the fourth edition of a university text in algebra, based on a course given in 1913. An able timely prologue defends the author's traditional taste while paying homage to the marked advances in synthesis during the first decades of this century and to the current notable shift toward structure concepts. The present volume falls into two parts: First Part — Classical Algebra in the Real and Complex Fields; Second Part — Theoretical Algebra in an Arbitrary Field. This second part is largely devoted to Galois theory with necessary preliminaries. There is a final chapter on the Axiomatics of the Algebraic Algorithm. This takes up, although very briefly, groups, rings, fields, domains of integrity, and the "Fundamental Theorem of Algebra" in the spirit of Gauss and his successors. A few exerciess are sacttered through the text. There are many effective historical and bibliographical notes. The author's enthusiasm and clarity of informal presentation should make the work welcome to Spanish undergraduates. Save for selection of teachable material and breadth of interest, no special originality need be claimed. *A. A. Bennett* (Providence, R.I.).

Combinatorial Analysis

Erdös, P.; and Rado, R. A partition calculus in set theory. Bull. Amer. Math. Soc. 62 (1956), 427–489.

"Dedekind's pigeon-hole principle, also known as the box argument or the chest of drawers argument (Schubfachprinzip) can be described, rather vaguely, as follows. If sufficiently many objects are distributed over not too many classes, then at least one class contains many of these objects. In 1930 F. P. Ramsey [Proc. London Math. Soc. (2) 30 (1929), 264–286] ··· discovered a remarkable extension of this principle which, in its simplest form, can be stated as follows. Let S be the set of all positive integers and suppose that all unordered pairs of distinct elements of S are distributed over two classes. Then there exists an infinite subset A of S such that all pairs of elements of A belong to the same class." The paper investigates a series of problems concerning more general "partition relations," involving order types as well as cardinals. The fundamental relation is

$$a \to (b_0, b_1)^2,$$

where a, b_0, b_1 are cardinals; it has the following meaning: whenever the set of all two-element subsets of a set S of power a is divided into two classes K_0 and K_1 (which may as well be disjoint), then either there is a subset B_0 of S of power b_0 all of whose two-element subsets belong to K_0, or there is a subset B_1 of S of power b_1 all of whose two-element subsets belong to K_1. Thus, for example, Ramsey's result is the assertion $\aleph_0 \to (\aleph_0, \aleph_0)^2$.

The following notational conventions are adopted: r, s denote nonnegative integers; $|S|$=cardinal of S; $[S]^r$=set of all subsets of S of power r; a, b, d denote cardinals; k, m, n, ν denote ordinals; α, β, γ, ϕ denote order types; $\alpha \leqq \beta$ signifies that a set of type β contains a set of type α. The order type of the linear continuum is denoted by λ. By definition, the general partition relation

$$a \to (b_0, b_1, \cdots)_k^r \tag{1}$$

holds if and only if b_ν is defined for all $\nu < k$, and whenever $|S| = a$, and $[S]^r$ is divided into classes K_ν $(\nu < k)$, then there exists $B \subset S$ and $\nu < k$ such that $|B| = b_\nu$ and $[B]^r \subset K_\nu$. If all b_ν are equal, say to b, then (1) is abbreviated to $a \to (b)_k^r$. Corresponding to (1),

$$\alpha \to (\beta_0, \beta_1, \cdots)_k^r \tag{2}$$

means that β_ν is defined for all $\nu < k$, and that whenever S is of type α, and $[S]^r$ is divided into classes K_ν $(\nu < k)$, then there exists $B \subset S$ and $\nu < k$ such that B is of type β_ν and $[B]^r \subset K$. The abbreviation $\alpha \to (\beta)_k^r$ has the obvious meaning.

Sections 1 and 2 of the paper are devoted to introductory remarks and definitions. Section 3 lists some previously known results, including the following. Theorem 1 (Ramsey): If k is finite, then $\aleph_0 \to (\aleph_0)_k^r$ (the result quoted above being a special case). Theorem 3: (i) (Dushnik-Miller-Erdös) $a \to (\aleph_0, a)^2$ for all infinite a; and (ii) (Erdös) $\aleph_\omega \nrightarrow (\aleph_1, \aleph_\omega)^2$ ($\nrightarrow$ signifying the denial of $\to$). Theorem 5 (Erdös-Rado): If ϕ is nondenumerable and $\phi \leqq \lambda$, $\alpha < \omega 2$, $\beta < \omega^2$, and $\gamma < \omega_1$, then $\phi \to (\omega, \gamma)^2$ and $\phi \to (\alpha, \beta)^2$. Theorem 6 (Erdös-Rado): $\eta \to (\aleph_0, \eta)^2$, where η denotes the order type of the set of rationals (this mixture of (1) and (2) having a fairly evident meaning). Theorem 9 (Sierpiński): If $|\phi| = |\lambda|$ and $\phi \leqq \lambda$, then $\lambda \nrightarrow (\phi, \phi)^1$. Theorem 10 (Erdös-Rado): For any a, $a \nrightarrow (\aleph_0, \aleph_0)^{\aleph_0}$ (with the obvious meaning for the infinite "exponent"; this result justifies the restriction in (1) and (2) to finite r).

Section 4 develops some preliminary results. Section 5 deals with denumerable order types. Theorem 23: If n is finite and $\alpha < \omega 2$, then $\omega n \to (n, \alpha)^2$, but $\omega n \nrightarrow (n+1, \omega+1)^2$. Theorem 24 (a special case of the somewhat complicated and difficult Theorem 25, not quoted here): If $\alpha < \omega 4$, then $\alpha \nrightarrow (3, \omega 2)^2$, but $\omega 4 \to (3, \omega 2)^2$. Section 6 is concerned with the type λ. Theorem 26 (ii): $\lambda \nrightarrow (r+1)_\omega^r$ for $r \geqq 2$. Theorem 29 (generalizing Theorem 9): If $2 \leqq |k| \leqq |\lambda|$, and $|\alpha_\nu| \geqq |\lambda|$ $(\nu < k)$, then $\lambda \nrightarrow (\alpha_0, \alpha_1, \cdots)_k^1$. Theorem 30: $|\lambda| \nrightarrow (\aleph_1, \aleph_1)^r$ for $r \geqq 2$.

Section 7 deals with arbitrary cardinals, and with $\aleph_1$ in particular. One of the main results of the paper is Theorem 31 (in particular, it implies Theorem 5): Let ϕ be a nondenumerable type containing neither ω_1 nor ω_1^*, and let $\alpha < \omega 2$, $\beta < \omega^2$, $\gamma < \omega\gamma_1$; then $\phi \to (\alpha, \alpha, \alpha)^2$, $\phi \to (\alpha, \beta)^2$, $\phi \to (\omega, \gamma)^2$, and $\phi \to (4, \alpha)^3$. The proof is very involved; at one stage, a diagram provided as an aid shows the inclusion relations and order relations among no less than twenty-five sets under consideration. Another difficult result is Theorem 33: If $\alpha < \omega 2$, then $\omega_1 \to (\alpha, \alpha)^2$. The somewhat complicated Theorem 34 (not quoted here) has the following consequences. Corollary 1: If $\aleph_n{}^d \leqq \aleph_n$ for all $d < \aleph_m$, then $\omega_{n+1} \to (\omega_m+1, \omega_{n+1})^2$. Corollary 2: If $\aleph_n$ is strongly inaccessible, then $\omega_n \to (\beta, \omega_n)^2$ for all $\beta < \omega_n$. A simple result is Theorem 36 (iii): $ab \nrightarrow (a^+, b^+)^2$ for any a, b (a^+ denoting the successor of a). Theorem 39 (ii): If $r > 0$, $\omega_n \to (\alpha_0, \alpha_1, \cdots)_k^r$, and $2\aleph_\nu \leqq \aleph_n$ $(\nu < n)$, then $\omega_{n+1} \to (\alpha_0+1, \alpha_1+1, \cdots)_k^{r+1}$. Theorem 41: If $r \geqq 3$, then for all n, $\omega_{n+1} \nrightarrow (\omega_n+2, \omega+1, r+1, \cdots, r+1)_{[r-1]}^r$. In particular, $\omega_1 \nrightarrow (\omega+2, \omega+1)^3$, whereas, by Theorem 39 (ii) and Theorem 1, $\omega_1 \to (\omega+1)_k^r$ for all r and all finite k. Theorem 43: $r < s \leqq \beta_0$, $\alpha \nrightarrow (\beta_0)_k^r$, and $\beta_1 \to (s)_k^r$, then $\alpha \nrightarrow (\beta_0, \beta_1)^s$ [a misprint has $\to$ for $\nrightarrow$ in this conclusion]. Some incidental consequences of this and other results are: $\omega_{n+1} \nrightarrow (\omega_1+1)_2{}^3$, where $\aleph_n = |\lambda|$; $|\lambda| \nrightarrow (\aleph_1, r+3)^r$ for $r \geqq 3$; and the rather amusing result $\omega_1 \nrightarrow (\omega+2, 2^{26}+r-4)^r$ for $r \geqq 4$.

Section 8 contains a theorem regarding the special concept of "canonical" partition relation. Section 9 is concerned with the more general idea of "polarised" partition relation. An instance of the latter is

$$\binom{a}{b} \to \binom{a_0 \quad a_1}{b_0 \quad b_1}^{r,s}$$

which means that whenever $|R| = a$ and $|S| = b$, and $[R]^r \times [S]^s$ is divided into classes K_0 and K_1, then there exists $\nu < 2$, and sets $A \subset R$ and $B \subset S$, such that $|A| = a_\nu$, $|B| = b_\nu$, and $[A]^r \times [B]^s \subset K_\nu$. A typical result is Theorem 48: The relation

$$\binom{\aleph_0}{b} \to \binom{\aleph_0 \quad \aleph_0}{\aleph_0 \quad b}^{1,1}$$

holds if and only if b is not expressible as the sum of a family of $\aleph_0$ cardinals each $< b$. In particular, the relation is valid for $b = \aleph_1$ and for $b = 2^{\aleph_0}$. *L. Gillman.*

Pólya, G. On picture-writing. Amer. Math. Monthly 63 (1956), 689–697.

The author convincingly shows that by stating a combinatory problem in terms of appropriate pictures which form a certain "figurate series" (this series constitutes a "complete survey of alternatives") and then replacing each picture by a suitable power of a certain variable, we are quite intuitively brought to introducing the generating function which ultimately provides the solution of our problem. — The method by which the figurate series is obtained is related to the well-known

construction of disjunctive normal forms in the sentential calculus; it might prove useful to go somewhat deeper into this relationship. *E. W. Beth* (Baltimore, Md.).

Roy, J.; and Laha, R. G. Two associate partially balanced designs involving three replications. Sankhyā **17** (1956), 175–184.

The authors give a complete enumeration of the designs in the title. The case $\lambda_1=1$, $\lambda_2=0$ has previously been treated by Bose and Clatworthy [Ann. Math. Statist. **26** (1955), 212–232; MR **17**, 227] It is interesting to note that the case $\lambda_1=2$, $\lambda_2=0$ is impossible. *H. B. Mann.*

Roy, J.; and Laha, R. G. Classification and analysis of linked block designs. Sankhyā **17** (1956), 115–132.

By interchanging blocks and varieties in a given incomplete block design, one obtains a new design called the dual of the original design. The dual of a balanced incomplete block design is called a linked block (LB) design; it has the property that any two blocks have λ elements in common. This paper gives the intrablock analysis for an LB design, its efficiency factor, and an exhaustive list of all LB designs with $k\leqq 10$, $r\leqq 10$. The necessary and sufficient condition for a partially balanced incomplete block design (PBIBD) to be an LB design is derived. The plans of some LB designs are given in detail. *R. G. Stanton* (Toronto, Ont.).

Ramakrishnan, C. S. On the dual of a PBIB design, and a new class of designs with two replications. Sankhyā **17** (1956), 133–142.

This paper gives the analysis and structure of the dual design [see the preceding review] of a PBIBD with two associate classes. In particular, the duals of some of these designs with two plots per block ($k=2$) are shown to be PBIBD's with 5 associate classes and $r=2$. The analysis of the designs is simpler by the dual method. *R. G. Stanton* (Toronto, Ont.).

Sprott, D. A. Some series of balanced incomplete block designs. Sankhyā **17** (1956), 185–192.

The author continues his work on the construction of Balanced Incomplete Block Designs [Canad. J. Math. **6** (1954), 341–346; MR **15**, 926]. The Bose Module Theorems are employed to give five 1 and 2 parameter families of designs. A typical result is the theorem: If

$$v=2am(2a\lambda+1)+1=p^n,$$

where p is a prime, and if among the $a\lambda$ exponents q_s, where

$$x^{2amq_s}-1=x^{q_0}\ (s=1, 2, \cdots, a\lambda),$$

each residue class (mod a) is represented λ times, then the design with parameters $v=2am(2a\lambda+1)+1$, $b=mv$, $r=m(2a\lambda+1)$, $k=2a\lambda+1$, λ, can be constructed from the initial blocks

$$(x^{ai}, x^{ai+4am}, \cdots, x^{ai+4a^2m\lambda})$$

where x is a primitive element of GF(v) and i ranges from 0 to $m-1$. *R. G. Stanton* (Toronto, Ont.).

See also: Čulík, p. 459

Čulík, K. Teilweise Lösung eines verallgemeinerten Problems von K. Zarankiewicz. Ann. Polon. Math. **3** (1956), 165–168.

Given positive integers i, j, m, n with $i\leqq m$, $j\leqq n$. Let $Z_{ij}(m, n)$ denote the least value of p having the following property: If a matrix with m rows and n columns has p elements zero and the remaining elements 1, then it has a zero submatrix with i rows and j columns. The determination of Z is a generalization of a problem of K. Zarankiewicz [Colloq. Math. **2** (1951), 301]. The author proves:

$$Z_{ij}(m, n)=(i-1)n+(j-1)\binom{m}{i}+1 \text{ for } n\geqq(j-1)\binom{m}{i}.$$

B. W. Jones (Boulder, Colo.).

Linear Algebra

★**Heller, I.; and Tompkins, C. B. An extension of a theorem of Dantzig's.** Linear inequalities and related systems, pp. 247–254. Annals o fMathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

Dantzig showed [Activity Analysis of Production and Allocation, Wiley, New York, 1951, pp. 359–373; MR **15**, 48] that the $(m+n)\times mn$ constraint matrix of the transportation problem has the special property that if any column is expressed as a linear combination of independent columns, the weights in the combination are 0, ± 1. It is here proved that if u_i, v_j ($i=1, \cdots, m$; $j=1, \cdots, n$) are a basis of a linear vector space of dimension $m+n$, then the following set S of vectors $(u_i, v_j, u_i+v_j, u_i-u_{i'}, v_j-v_{j'})$ ($i, i'=1, \cdots, m$; $j, j'=1, \cdots, n$) has the same property, i.e. that if any vector s of S is represented as a linear combination of vectors in a basis culled from S, the weights take only the values 0, ± 1. The columns of the transportation matrix are obtained as the subset (u_i+v_j) of S, for the case that u_i, v_j form an orthonormal system.

The relevance for linear programming is via the existence of integer-valued solutions to a linear program constrained by $Ax=b$. If A has the Dantzig property, either all basic representations of b in A are integral or none are (and no non-basic ones exist either), so that the existence of one solution (basic or not) with integral coordinates is sufficient to assure that all basic solutions (and hence all optimal ones) are integral. *R. Solow.*

Hodges, John H. Exponential sums for symmetric matrices in a finite field. Math. Nachr. **14** (1955), 331–339 (1956).

Let $q=p^n$, p odd; for $\alpha\in$ GF(q) define

$$e(\alpha)=e^{2\pi i t(\alpha)/p},\ t(\alpha)=\alpha+\alpha^p+\cdots+\alpha^{p^{n-1}}.$$

If $A=(\alpha_{ij})$ is a symmetric matrix, put $\sigma(A)=\sum\alpha_{ii}$. Also if $U'AU=\text{diag}(\alpha_1, \cdots, \alpha_r, 0, \cdots, 0)$, U non-singular put $\lambda(A)=\psi(\delta(A))$, where $\delta(A)=\alpha_1\alpha_2\cdots\alpha_r$ and $\psi(\alpha)=0, +1, -1$ according as α is zero, a square or a non-square of GF(q). $\lambda(A)$ is called the invariant of A; $A(m, r, \lambda)$ denotes a symmetric matrix of order m, rank r and invariant λ.

The sum

$$H(B, z, \lambda)=\sum_{C(q,s,\lambda)} e\{-\sigma(BC)\},$$

was defined by the reviewer [Duke Math. J. **21** (1954), 123–137; MR **15**, 604]. The principal object of the present paper is to evaluate this sum explicitly. It is convenient to define

$$S(B, z)=H(B, z, 1)+H(B, z, -1),$$
$$D(B, z)=H(B, z, 1)-H(B, z, -1).$$

The functions $S(B, z)$ and $D(B, z)$ are evaluated in every case. For example, if B is of rank $2s$, then

$$D(B, 2j+1)=0 \ (j\geqq 0),$$

$$D(B, 2j)=(-1)^j q^{-j} \sum_{k=0}^{j} q^{k(k+1)+2s(j-k)} \begin{bmatrix} s \\ k \end{bmatrix} \frac{\prod_{i=0}^{s(j-k)-1}(1-q^{i-2s-j})}{\prod_{i=1}^{j-k}(1-q^{-2i})} \ (j\geqq 1),$$

where

$$\begin{bmatrix} s \\ k \end{bmatrix}=\frac{(1-q^{2s})(1-q^{2s-2})\cdots(1-q^{2s-2k+2})}{(1-q^2)(1-q^4)\cdots(1-q^{2k})}.$$

Also the functions $S(B, z)$, $D(B, z)$ are exhibited in terms of terminating q-hypergeometric functions of type ${}_2\Phi_2$.

The sum

$$T(A)=\sum \lambda(B)e(-\sigma(AB))$$

where the summation is over all symmetric B, can be expressed in terms of $D(A, z)$ and hence is known explicitly.

Application of the above results is made to find the number of $m\times t$ matrices X such that

$$X'AX=B+Y,$$

where $A=A(m, m)$, B is of order t and $Y=Y(t, s, \lambda)$; also to find the number of ways of partitioning a given symmetric matrix of order t into the sum of k symmetric matrices of order t having rank r_i and invariant λ_i for $i=1, \cdots, k$. *L. Carlitz* (Durham, N.C.).

Mirsky, L. The norms of adjugate and inverse matrices. Arch. Math. 7 (1956), 276–277.

The author gives a very simple proof of an inequality of H. Richter [Arch. Math. 5 (1954), 447–448; MR 16, 106]. The assertion is that $\|\text{adj } A\|\leqq n^{-(n-2)/2}\|A\|^{n-1}$, where A is an n-rowed square matrix (complex), $\|A\|^2=\text{trace}(A^*A)$, and adj A is the "adjugate" of A (so that $A^{-1}=\text{adj } A/\det A$ whenever $\det A\neq 0$). The proof uses the polar decomposition theorem and an easy inequality involving the elementary symmetric functions of a set of non-negative numbers. *P. R. Halmos* (Chicago, Ill.).

Brenner, J. L. Neuer Beweis eines Satzes von Taussky und Geiringer. Arch. Math. 7 (1956), 274–275.

The theorem in question states that an $n\times n$ matrix $A=(a_{ik})$ with $|a_{ii}|\geqq\sum_{k\neq i}|a_{ik}|$ $(i=1, \cdots, n)$ is non-singular provided that at least one strict inequality occurs and that A is indecomposable. The new proof is based on a lemma concerning the minors of such a matrix.

In the statement of the lemma replace "grösser als" by "mindestens so gross wie". *M. Marcus.*

Mirsky, L. The spread of a matrix. Mathematika 3 (1956), 127–130.

Let the complex $n\times n$ matrix $A=(a_{ik})$ have characteristic roots $\omega_1, \cdots, \omega_n$ $(n\geqq 3)$. Set $s(A)=\max_{i,k}|\omega_i-\omega_k|$, and $s_R(A)=\max_{i,k}\{\Re\omega_i-\Re\omega_k\}$; $s(A)$ is called the spread of A. The author shows that

$$s(A)\leqq\{2\|A\|^2-\frac{2}{n}|\text{tr } A|^2\}^{1/2},$$

where $\|A\|^2=\text{tr } A^*A$, with equality if and only if A is normal and $n-2$ of its characteristic roots are equal to each other and to the arithmetic mean of the remaining two. Also,

$$s_R(A)\leqq\{\|A\|^2+\Re\text{ tr }(A^2)-\frac{2}{n}(\Re\text{ tr } A)^2\}^{1/2},$$

with a similar condition for equality. If A is normal, $s(A)\geqq 3^{1/2}\max_{i\neq k}|a_{ik}|$, and $s_R(A)\geqq\max_{i\neq k}|a_{ik}+\bar{a}_{ki}|$.

B. N. Moyls (Vancouver, B.C.).

See also: Rosenberg, p. 462; Schönberg, p. 462; Borg, p. 474; Širokov, p. 508; Householder, p. 514; Aoyama, p. 515; Charnes and Miller, p. 547.

Polynomials

Kegel, Günter. On the roots of polynomials defined by recurrence formulae. An. Acad. Brasil. Ci. 28 (1956), 165–178. (Portuguese summary)

Polynomials $P_n(x)$ which satisfy recurrence relations of the form

$$P_0(x)=A_0,\ P_1(x)=(A_1x+B_1)P_0(x),$$

$$P_n(x)=(A_nx+B_n)P_{n-1}(x)-C_nP_{n-2}(x) \ (n\leqq 2)$$

have frequently been discussed in different contexts, e.g., in the theory of orthogonal polynomials and in connection with continued fractions. When $A_nA_{n-1}C_n>0$, the roots of $P_n(x)$ are real and distinct and are separated by those of $P_{n-1}(x)$. Assuming this condition to hold the author finds upper and lower bounds for the roots of $P_n(x)$. As the numerical determination of these bounds is likely to be difficult except in simple cases, a graphical method is recommended to obtain approximations. The remarks are illustrated by applications to the Hermite and Laguerre polynomials. *W. Ledermann* (Manchester).

Corson, H. H. On some special systems of equations. Pacific J. Math. 6 (1956), 449–452.

Let F be an arbitrary field. Consider the system

$$(*) \qquad x_1^{k_1}=f(x_3, \cdots, x_n),\ x_2^{k_2}=g(x_3, \cdots, x_n),$$

also the equation

$$(**) \qquad y^{k_1k_2}=f^{sk_2}(y_3, \cdots, y_n)g^{rk_1}(y_3, \cdots, y_n),$$

where $(k_1, k_2)=1$ and $rk_1+sk_2=1$. Then it is proved that the distinct solutions of (*) in F with $x_1x_2\neq 0$ may be put in a one-to-one correspondence with the distinct solutions of (2) with $y\neq 0$. As an application, it is shown that if f and g are homogeneous polynomials of respective degree m_1, m_2, where $(m_1, k_1)=1=(m_2, k_2)$, then the total number of solutions of (*) in GF(q) is q^{n-2}. Also for the system

$$x_1^{k_1}=a_3x_3^{em_3}+\cdots+a_nx_n^{em_n},$$

$$x_2^{k_2}=b_3x_3^{dm_3}+\cdots+b_nx_n^{dm_n},$$

where $(k_1, k_2)=1$, $M=\text{l.c.m.}(m_3, \cdots, m_n)$, $(eM, k_1)=(dM, k_2)=1$, the number of solutions is q^{n-2}. A like result holds when $x_i^{em_i}$ is replaced by $f_i(x_i)$ and $x_i^{dm_i}$ is replaced by $g_i(x_i)$.

Finally, if f and g are homogeneous polynomials with integral coefficients of degree m_1, m_2 where $(m_1, k_1)=(m_2, k_2)=1$, a family of integral solutions of (1) is obtained. However, this does not in general furnish all the integral solutions. *L. Carlitz* (Durham, N.C.).

Ostrowski, Alexander. Über die Darstellung von symmetrischen Funktionen durch Potenzsummen. Math. Ann. **132** (1956), 362–372.

The classic subject developed by Isaac Newton and Jacques Bernoulli relating elementary symmetric functions and sums of powers has interested numerous algebraists ever since and is a standard topic in undergraduate texts in Algebra. It is surprising therefore that this author seems to be the first to note the interesting simple fact that in expanding any elementary symmetric function as a polynomial in sums of powers, while the coefficients have algebraic sum zero, their absolute values have sum unity. Not less surprising is the fact that for the case of a finite combination with repeated elements, first studied by Cayley, and since discussed repeatedly in standard texts, there have been errors throughout in making the count of coefficients, due to the mistaken notion that a simple limiting operation reducing previously distinct elements into mutual replicas should yield the desired enumeration. Correct (but somewhat complicated) expressions are given here for the algebraic sum, and for the sum of absolute values of the coefficients of the expansion of a given sum of powers as a polynomial in terms of elementary symmetric functions. The discussion covers also the unreduced function type:

$$[p_1, \cdots, p_r]=\sum x_1^{p_1}\cdots x_r^{p_r},$$

and Cayley's corresponding $(p_1, \cdots, p_r)$, which is the sum of the mutually distinct terms of $[p_1, \cdots, p_r]$. The methods of proof used are elementary and largely traditional. *A. A. Bennett* (Providence, R.I.).

Neville, E. H. Schur's inequality and Watson's identities. Math. Gaz. **40** (1956), 288–291.

The author establishes, by straightforward but hardly obvious methods, various identities for the symmetric function

$$\varphi\equiv x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y),$$

including those found by Watson [Math. Gaz. **37** (1953), 244–246; **39** (1955), 207–208; MR **15**, 407; **17**, 134] in connection with the fact that $\varphi\geqq 0$ for nonnegative x, y, z. He shows that φ is not expressible as a sum of nonnegative products. *R. P. Boas, Jr.* (Evanston, Ill.).

See also: Mirsky, p. 460; Lenz, p. 501; Fiedler, p. 515.

Theory of Invariants

★**Lemmon, William W. Quaternion invariants of the Minkowski space.** Navy Contract N7–ONR–434, Task Order 3. Technical report, Dept. of Math., Tulane Univ., New Orleans, Louisiana, 1955. iii+68 pp.

Partial Order Structures

Climescu, Al. Quelques observations sur les semistructures d'ordre fini. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi **2** (1955), 1–4. (Romanian. Russian and French summaries)

A semistructure (or semilattice) is an algebra with one binary associative commutative operation for which every element is idempotent. There are twelve "observations" of which I–VI show the existence of six types of subalgebra, and the rest lead to XII: The set of ideals of a semistructure of order n, partially ordered by inclusion, is a distributive lattice of length $n-1$.

(In IV, line 5, for $ay=y$ read $ay=a$; in XI, line 3, for $a=b$ read $ab=b$.) *I. M. H. Etherington* (Edinburgh).

Dwinger, Ph. On the lattice of the closure operators of a complete lattice. Nieuw Arch. Wisk. (3) **4** (1956), 112–117.

The author considers the set C_L of all closure operations φ of a complete lattice L. These φ may be partially ordered under the natural inclusion and it is known that C_L is a complete lattice. The main result of the paper is that C_L is modular if and only if L is a chain. Certain special cases where C_L is distributive are also examined. *O. Ore.*

Edmondson, Don E. A nonmodular compact connected topological lattice. Proc. Amer. Math. Soc. **7** (1956), 1157–1158.

The author constructs a pair of continuous lattice operations on the closed 3-cell in such a way that the condition of modularity is not satisfied. *A. D. Wallace.*

Dwinger, Ph.; and de Groot, J. On the axioms of Baer and Kurosh in modular lattices. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 596–601.

The modular axiom may be considered a weak form of the distributive law in lattices. Two infinite modular laws have been introduced by Baer [Trans. Amer. Math. Soc. **64** (1948), 519–551; MR **10**, 425] and Kurosh [Izv. Akad. Nauk SSSR. Ser. Mat. **7** (1943), 185–202; MR **6**, 145]. The authors show that the axiom of Baer is stronger than that of Kurosh. *O. Ore* (New Haven, Conn.).

Ehrlich, Gertrude. Characterization of a continuous geometry within the unit group. Trans. Amer. Math. Soc. **83** (1956), 397–416.

Aside from improvements in presentation this paper is essentially the thesis of the authoress [Univ. of Tennessee, 1953; MR **15**, 930]. The important Theorem 1 which characterizes elements of class 2 is attributed to W. Givens. The theorem that the centre of eRe is eZe if R is a continuous ring with centre Z and e is idempotent, attributed to von Neumann (unpublished manuscript), was later found independently by W. Givens; the proof is not given in the present paper but the reviewer points out that this theorem can be avoided in a more general argument. *I. Halperin* (Kingston, Ont.).

See also: Jaffard, p. 464; Isbell and Rubin, p. 495.

Rings, Fields, Algebras

Krull, Wolfgang. Über reelle Radikalkörper. Math. Z. **65** (1956), 76–90.

Let $K_i=K_{i-1}(\alpha_i)$ $(i=1, 2, \cdots, n)$ be a chain of fields (all contained in the field of real numbers), where each α_i is a root of an irreducible polynomial $x^{p_i}-a_i$, $a_i \in K_{i-1}$, p_i prime. Any subfield $K\supset K_0$ of such a K_n is called a real radical field over K_0 (r.r. field). The classical p-conjugate theorem states that if the degree of a r.r. field K over its base field K_0 is a prime $p\neq 2$, then all the proper conjugate fields of K over K_0 are complex. In this paper the author first characterizes such r.r.fields of prime degree. We shall not state his general result. For Gaussian primes,

however, the result is as follows: If $K_0(\alpha)$ is a real metacyclic extension of K_0 such that $[K_0(\alpha):K_0]$ is a prime of the form 2^m+1, then $K_0(\alpha)$ is a r.r.field over K_0 if and only if the smallest normal extension of K_0 containing $K_0(\alpha)$ is complex.

In order to generalize the p-conjugate theorem, the author introduces the notion of a maximal r.r.field. K is a maximal r.r.field over K_0 if it is the union of all r.r.fields contained in the smallest normal extension of K_0 which contains K. Each such K can be obtained by a chain of fields $K_0 \subset K_1 \subset \cdots \subset K_s = K$, where $[K_i:K_{i-1}]$ is a prime. Using this chain and the theory of r.r.fields of prime degree, the author shows that all the proper conjugate fields of a maximal r.r.field are complex. He gives an example to show that there exist r.r.fields (with prime chains) having proper conjugate fields which are real and yet not r.r.fields. The number of automorphisms of a maximal r.r.field is always a power of 2. *B. N. Moyls.*

Rédei, Ladislaus. Äquivalenz der Sätze von Kronecker-Hensel und von Szekeres für die Ideale des Polynomringes einer Unbestimmten über einem kommutativen Hauptidealring mit Primzerlegung. Acta Sci. Math. Szeged **17** (1956), 198–202.

Let R be a commutative principal ideal ring and let $\mathfrak{R}$ denote the classes of associated non-zero numbers of R. The writer shows that the theorem of Szekeres [Amer. Math. Monthly **59** (1952), 379–386, p. 385; MR **13**, 903] can be put in the following form. Let $0=n_0<n_1<\cdots<n_r$ ($r\geqq 0$) be rational integers, $\sigma_1, \cdots, \sigma_r \in \mathfrak{R}$, $f_{kl}(x)$ ($1\leqq k\leqq l\leqq r$) polynomials with coefficients $\in \mathfrak{R}(\sigma_i)$ and of degree $<n_k-n_{k-1}$. Determine the polynomials $F_0(x), \cdots, F_r(x)$ recursively by means of

$$F_0(x)=\sigma_1\cdots\sigma_r,$$

$$\sigma_1 F_1(x)=[x^{n_1}+f_{11}(x)]F_0(x),$$

$$\sigma_2 F_2(x)=f_{12}(x)F_0(x)+[x^{n_2-n_1}+f_{22}(x)]F_1(x),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sigma_r F_r(x)=f_{1r}(x)F_0(x)+f_{2r}(x)F_1(x)+\cdots+[x^{n_r-n_{r-1}}+f_{rr}(x)]F_{r-1}(x).$$

Then the ideals of $R[x]$ are given by

$$\mathfrak{a}=(F_0(x), \cdots, F_r(x)).$$

This result is a generalized version of the theorem of Kronecker [Vorlesungen über Zahlentheorie, Bd. 1, Teubner, Leipzig, 1901, Lecture 18]. *L. Carlitz.*

Northcott, D. G. On the notion of a first neighbourhood ring with an application to the $AF+B\Phi$ theorem. Proc. Cambridge Philos. Soc. **53** (1957), 43–56.

Soit Q un anneau local de dimension 1 dont l'idéal maximal $\mathfrak{m}$ ne se compose pas uniquement de diviseurs de zéro. Etude des éléments superficiels pour $\mathfrak{m}$, et construction du premier anneau de voisinage R de Q: c'est l'ensemble des fractions a/b où $a \in \mathfrak{m}^s$, $b \in \mathfrak{m}^s$ et où b est superficiel de degré s pour $\mathfrak{m}$; ceci généralise la définition donnée dans le cas des anneaux d'intégrité [Northcott, J. London Math. Soc. **30** (1955), 360–375; MR **17**, 86.] L'anneau R est un Q-module de type fini. Pour que $R=Q$ il faut et il suffit que Q soit régulier. Etude de R au moyen d'un anneau de polynômes auxiliaire. Application au „théorème $AF+B\Phi$" dans un anneau local régulier de dimension 2. *P. Samuel* (Clermont-Ferrand).

Rosenberg, Alex. The Cartan-Brauer-Hua theorem for matrix and local matrix rings. Proc. Amer. Math. Soc. **7** (1956), 891–898.

Let A be a ring with unit and let A_n be the ring of $n\times n$ matrices over A. Let J be any nonzero two sided ideal of A, and let J_n denote the ring of $n\times n$ matrices with elements belonging to J. Let e_{ij} denote a matrix with zero elements everywhere except 1 at (ij) position. The matrix conjugate with $I+\lambda e_{ij}$ $(i\neq j)$ is called a transvection. We use $[A_n, J_n]$ to denote the additive group generated by elements of the form $ab-ba$ with $a \in A_n$, $b \in B_n$. Let D denote an additive group of diagonal matrices $\sum \delta_i e_{ii}$, where $\delta_i \equiv \delta_j \pmod J$ and $\delta_i + J$ is in the center of $A-J$.

Evidently, the set $S=[A_n, J_n]+D$ forms an additive group and for any transvection t and every element of $a \in S$, tst^{-1} belongs also to S. The converse of this theorem was proved, for $n\geqq 3$. The case $n=2$ has been also studied. Similar results have been obtained for the local matrix ring. *L. K. Hua* (Peking).

Schönberg, Mario. On the Grassmann and Clifford algebras. I. An. Acad. Brasil. Ci. **28** (1956), 11–19.

This is the first one of a series of papers in which the author is to discuss certain geometric algebras and to give applications to physical theories, especially quantum mechanics. Use is made of the notions and notations of the tensor calculus. The associative algebras discussed in this paper are mainly the two types defined over the field F of real or complex numbers by $2n$ generators I_j $(j=1, \cdots, n)$ and I^j $(j=1, \cdots, n)$, with identity 1, subject to the rules

$$I_j \cdot I_k \pm I_k \cdot I_j = 0,\quad I^j \cdot I^k \pm I^k \cdot I^j = 0,$$

$$I^j \cdot I_k \pm I_k \cdot I^j = \delta_k^j 1,$$

and denoted by G_n (for the plus signs) and L_n (for the minus signs). Both are extensions of Grassmann algebras if the $a^j I_j$ and $a_j I^j$ (a^j, $a_j \in F$) are interpreted as contravariant and covariant vectors resp. of a n-dimensional affine space over F. G_n and (for even n) L_n are shown to be Clifford and symplectic algebras resp. of a $2n$-dimensional space. L_n over the complex numbers is the Heisenberg algebra of the position and momentum operators with n degrees of freedom. *A. Jaeger* (Wuerzburg).

Drazin, M. P. Corrections to the paper "Engel rings and a result of Herstein and Kaplansky". Amer. J. Math. **78** (1956), 224.

Drazin, M. P. Corrections to "Engel rings and a result of Herstein and Kaplansky." Amer. J. Math. **78** (1956), 899.

The author points out errors and gaps in some of the theorems in a previous paper [same J. **77** (1955), 895–913; MR **17**, 1048] and points out how these affect the remaining results of the paper. *I. N. Herstein.*

Yoshii, Tensho. On algebras of bounded representation type. Osaka Math. J. **8** (1956), 51–105.

Yoshii, Tensho. Note on algebras of bounded representation type. Proc. Japan Acad. **32** (1956), 441–445.

Let A be an associative algebra with unit element and of finite dimension over a field K. A is said to be of bounded representation type when every directly indecomposable representation of A in K has a degree less than a fixed finite value. The first paper gives a necessary and sufficient condition for A to be bounded under the assumption that K is algebraically closed and the radical

N of A satisfies $N^2=0$. Thus, let e_i denote, generally, a primitive idempotent in A. $(e_1, e_2, \cdots, e_t)$ is said to form a chain if for each $i=1, \cdots, t-1$ we have $e_i \neq e_{i+1}$ and Ne_i, Ne_{i+1} have isomorphic simple left subideals. Further, if e_1, e_2, e_3 are nonisomorphic to one another and if Ne_1, Ne_2, Ne_3 have isomorphic simple left subideals, then we say that e_1 is divided, and indeed divided into e_2, e_3. Now, if K is algebraically closed and $N^2=0$ then the following set of conditions is shown to be necessary and sufficient for A to be of bounded representation type:

1) There is no (primitive idempotent) e_i such that Ne_i has two distinct mutually isomorphic simple left subideals.

2) There is no e_i such that Ne_i contains a direct sum of four simple left subideals.

3) There is no chain $(e_1, \cdots, e_t)$ with $t \geqq 2$, $e_1 \simeq e_t$.

4) If $(e_1, \cdots, e_t)$, $t \geqq 2$, is a chain and if either Ne_t is a direct sum of three simple left subideals or e_t is divided, then Ne_1 is not a direct sum of three simple left subideals.

5) If each of Ne_1, Ne_2, Ne_3 is a direct sum of three simple left subideals then e_1 is not divided into e_2, e_3.

6) If $(e_1, \cdots, e_t)$ is a chain and if some Ne_i with $i \neq 1, t$ is a direct sum of three simple left subideals, then either $t=4$ and Ne_1, Ne_4 are simple left ideals, or $t=3$.

7) The right-left symmetries to 1)–6) hold.

{It seems to the reviewer that in 2) it is assumed that for $i=2, \cdots, t-1$, Ne_{i-1} and Ne_{i+1} do not possess simple left subideals isomorphic to each other.} The proof consists, as is perhaps expected, of many complicated computations and case divisions.

The second paper shows then that under the above circumstances A has indeed only a finite number of inequivalent directly indecomposable representations in K, (a part of) a conjecture of R. Brauer and R. Thrall.

T. Nakayama (Nagoya).

Hall, Marshall, Jr. An identity in Jordan rings. Proc. Amer. Math. Soc. **7** (1956), 990–998.

Suppose that in a Jordan ring $\{abc\}$ denotes the Jordan triple product. It had been known that in a special Jordan ring $\{aba\}^2=\{a\{ba^2b\}a\}$, and here Hall deduces this relation in Jordan rings by presenting generators and relations in the free Jordan ring on 2 generators up through some of those of degree 6. His computations bolster the belief that this Jordan ring is special. Just as Hall's paper was being published A. I. Shirshov [Mat. Sb. N.S. **38(80)** (1956), 149–166; MR **17**, 822] turned this conjecture into fact. *W. G. Lister* (Providence, R.I.).

Levin, Frank. On ideals in multidifferential polynomial rings. Math. Ann. **132** (1956), 289–309.

The author extends to the partial case certain results and methods from the theory of ordinary linear differential polynomial rings as given by Ore [J. Reine Angew. Math. **167** (1932), 221–234]. Let F be a MD-(multidifferential) field, i.e. a field with a finite set D of derivations D_i $(i=1, \cdots, n)$ commuting in pairs, of characteristic zero. D is called regular if the D_i are linearly independent in the F-module of all derivations of F. The set $\Omega=F\{U\}$ of all linear MD-polynomials $f(U), g(U), \cdots$ forms a ring if addition is defined in the usual way and multiplication by substitution: $(fg)(U)=f(g(U))$. The adjoint f^* of a MD-polynomial f is introduced and $f \to f^*$ is seen to be an anti-automorphism of Ω. The linear MD-polynomial ring Ω is shown to be simple if and only if D is regular. Now left ideals are investigated. A constructive proof is given that each ideal Φ of Ω has a finite basis, which includes an estimate of the number of basis elements and a necessary and sufficient condition that Φ is principal. The ideal generated by the set $\{D_iU+a_iU\}$ $(i=1, \cdots, n$; all a_i constants) is maximal, so is the ideal Φ_a generated by the set $\{D_iU-a^{-1}(D_ia)U\}$ $(i=1, \cdots, n$; $a(\neq 0) \in F)$ which is itself a canonical basis for Φ_a. The linear differential equation $f(U)=b$ has the solution $a \neq 0$ if and only if $f=a^{-1}b \pmod{\Phi_a}$. Let $\langle f \rangle$ be the principal ideal generated by the MD-polynomial f, and let (Φ, Ψ) and $[\Phi, \Psi]$ be the ideals generated by the set union and intersection resp. of the ideals Φ and Ψ. Then an ideal Φ is said to be G-similar to an ideal Ψ if there exists a MD-polynomial f in Ω such that (*) $[\langle f \rangle, \Psi]=f\Phi$ holds, and it is said to be R-similar if besides (*) also $(\langle f \rangle, \Psi)=\langle 1 \rangle$ holds. The author shows that R-similarity is an equivalence relation and thereby that in the Noether-Schmeidler definition of similarity [Math. Z. **8** (1920), 1–35] the added assumption of symmetry is superfluous. He then gives elegant proofs of a number of theorems on R-similarity, for example: If an ideal Φ can be written in two different ways as a finite intersection of maximal ideals, then each ideal in the first representation is R-similar to one in the second one. Finally some results on G-similarity are reported. *A. Jaeger* (Wuerzburg).

Seidenberg, A. Contribution to the Picard-Vessiot theory of homogeneous linear differential equations. Amer. J. Math. **78** (1956), 808–818.

Let F be an ordinary differential field of characteristic 0 and field of constants C, and let $L(y)=\sum_{i=0}^{n} p_i y^{(n-i)}$ (each $p_i \in F$, $p_0=1$). It is known [Kolchin, Bull. Amer. Math. Soc. **54** (1948), 927–932; MR **10**, 349] that there exists a fundamental system $\eta_1, \cdots, \eta_n$ of solutions of $L(y)=0$ such that the field of constants D of $F\langle \eta_1, \cdots, \eta_n \rangle$ is algebraic over C, and even [M. P. Epstein, Proc. Amer. Math. Soc. **6** (1955), 33–41; MR **16**, 670] normal over C. The author gives a counterexample for $n=2$, showing that one cannot always choose the fundamental system to obtain $D=C$. More generally, he gives a complete analysis of the case $n=2$ as follows: One can obtain $D=C$ if and only if the associated Riccati equation has a solution α such that the field of constants C_α of $F\langle \alpha \rangle$ is C; if η is a generic solution of the Riccati equation, C_η is either C or of transcendence degree 1 and genus 0 over C; in the latter case there exists an α with $C_\alpha=C$ if and only if C_η is a simple extension of C. *E. R. Kolchin.*

Bartholomay, Anthony F. The Serre group $E_2^{p,q}$. Portugal. Math. **15** (1956), 31–34.

Let A be a graded differential group with a filtration. The group $E_2^{p,q}$ of the spectral sequence of A, defined by Serre [Ann. of Math. (2) **54** (1951), 425–505, p. 431; MR **13**, 574] is shown in this paper to be equivalent to $\hat{E}_2^{p,q}=(\text{image } f)$, where

$$f: H_{p+q}(A^p/A^{p-2}) \to H_{p+q}(A^{p+1}/A^{p-1})$$

is the map

$$(i \circ j'')_*: H_{p+q}(A^p/A^{p-2}) \to H_{p+q}(A^p/A^{p-1}) \to H_{p+q}(A^{p+1}/A^{p-1})$$

induced on the homology groups by the maps associated with the inclusions among the (A^p).

T. R. Brahana.

★ **Zelinsky, Daniel. Cohomology of function fields and other algebras.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 227–231. Science Council of Japan, Tokyo, 1956.

Account of results obtained jointly with A. Rosenberg [Trans. Amer. Math. Soc. **82** (1956), 85–98; MR **17**, 1181].

G. P. Hochschild (Princeton, N.J.).

★ **Inaba, Eizi. On cohomology groups in a field, which is complete with respect to a discrete valuation.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 238–239. Science Council of Japan, Tokyo, 1956.

Abstract of a paper published elsewhere [Nat. Sci. Rep. Ochanomizu Univ. **6** (1955), 25–29; MR **17**, 1231].

G. P. Hochschild (Princeton, N.J.).

See also: Jaffard, p. 464; Oganesyan, p. 465; Vandiver and Weaver, p. 465; Nöbauer, p. 466; Schwarz, p. 467; Kuranishi, p. 474; Munn, p. 489; Curtis, p. 491; Ree, p. 491; Vasilach, p. 491; Al-Dhahir, p. 502; Pickert, p. 502; Rosati, p. 512.

Groups, Generalized Groups

Green, J. A. On the number of automorphisms of a finite group. Proc. Roy. Soc. London. Ser. A. **237** (1956), 574–581.

The main result is the following refinement of a result due to W. Ledermann and B. H. Neumann [same Proc. **233** (1956), 494–506; MR **17**, 580]: let G be a finite group, p a prime. Then the order of the automorphism group of G is divisible by p^h, provided only that the order of G is divisible by p^g, where $g=\frac{1}{2}h(h+3)+1$. The proof results from a refinement of the procedure followed by Ledermann and Neumann and the use of improved estimates for the orders of certain auxiliary groups.

G. P. Hochschild (Princeton, N.J.).

Łoś, J. On the torsion-free Abelian groups with hereditarily generating sequences. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 169–171.

In Publ. Math. Debrecen **4** (1956), 351–356 [MR **18**, 190] J. Łoś, E. Sąsiada and Z. Słomiński call a sequence of elements of an abelian group G an hereditarily generating sequence (h.g.s.) if each infinite subsequence generates G. In addition to classifying the torsion groups with h.g.s., they prove that any abelian group with h.g.s. has (I) finite rank, and (II) no finite and non-cyclic homomorphic images. In the present paper the author proves that for torsion free abelian groups, (I) and (II) are sufficient as well as necessary for the existence of an h.g.s. The proof relies on a theorem of B. H. Neumann [J. London Math. Soc. **29** (1954), 236–248; MR **15**, 931] from which it is inferred that (II) is equivalent to the condition that G is not a set theoretic union of a finite number of its proper subgroups. *D. G. Higman.*

Sąsiada, E. An application of Kulikov's basic subgroups in the theory of Abelian mixed groups. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 411–413.

Let G be an abelian group, T its torsion subgroup, $H=G/T$. (1) Let T and H be fixed. For it always to be true that T is a direct summand, it is necessary and sufficient that this hold for the basic subgroup of T. (2) Let just H be fixed. To know that H always makes T a direct summand, it suffices to test it on the case where T is a direct sum of cyclic groups. *I. Kaplansky.*

Blackburn, N. On prime-power groups in which the derived group has two generators. Proc. Cambridge Philos. Soc. **53** (1957), 19–27.

It is shown that if the derived group G' of a finite p-group G is not Abelian, and if G' can be generated by two elements a and b, then a and b can be chosen so that the defining relations of G' are $a^{p^m}=b^{p^{m+k}}=1$, $[a,b]=b^{p^k}$, where $0<2k\leq m\leq n$. Conversely, every such group occurs as the derived group of a suitable finite p-group. As applications, it is shown that if G is a finite p-group such that both G and G' can be generated by two elements, then G' is Abelian; and that a group of order 3^n and of class $n-1$ is metabelian. *P. Hall.*

★ **Jaffard, P. Extensions des groupes ordonnés.** Séminaire A. Châtelet et P. Dubreil de la Faculté des Sciences de Paris, 1953/1954. Algèbre et théorie des nombres. 2e tirage multigraphié, pp. 11-01—11-10. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956.

The author solves the following extension problem for (partially) ordered abelian groups: Given two ordered groups H and Q and an extension G of H by Q. Find all prolongations of the ordering of H to G such that H becomes an isolated subgroup of G and the ordering induced in $Q=G/H$ coincides with the given one. The order structure of G is discussed, necessary and sufficient conditions for G_+ are given. Let the elements of G be represented in the usual way as elements of the product set $H\times Q$, then G is called a lexicographical extension if:

$$(a,x)\geqq 0 \leftrightarrow \{x>0 \text{ or } x=0 \text{ and } a\geqq 0\}.$$

A lexicographical extension of $H\neq\{0\}$ by Q is lattice-ordered if and only if H is lattice-ordered and Q totally ordered. A lattice-ordered group is called by the author of class 0 if $G=\{0\}$, and of class n (n a positive integer) if it is a lexicographical extension of a lattice-ordered group of class $n-1$ by a totally ordered group or if it is the ordered direct sum of finitely many such extensions. In remaining part of the paper some applications to abelian lattice-ordered groups having only finitely many filets [Jaffard, C. R. Acad. Sci. Paris **230** (1950), 1024–1025, 1125–1126; J. Math. Pures Appl. (9) **32** (1953), 203–280; MR **11**, 579; **15**, 284] are given. Every such group is proved to belong to one of the above classes, and it is shown how lattice-ordered groups may be decomposed uniquely into ordered direct sums of lattice-ordered groups. *A. Jaeger* (Wuerzburg).

Jaffard, Paul. Sur la théorie algébrique de la croissance. C. R. Acad. Sci. Paris **243** (1956), 1383–1385.

$\mathfrak{G}$ sei die additive Verbandsgruppe der reellen Funktionen $f, g, \cdots$ auf einer Menge E und $\mathfrak{F}$ ein Filter von E. $\mathfrak{H}_\mathfrak{F}$ sei die Untergruppe der f mit $E(f=0)\in\mathfrak{F}$, $\mathfrak{G}_\mathfrak{F}$ sei die Faktorgruppe $\mathfrak{G}$ mod $\mathfrak{H}_\mathfrak{F}$. Nach W. Krull [Math. Z. **64** (1956), 10–40; MR **17**, 582] ist $\mathfrak{G}_\mathfrak{F}$ genau dann total geordnet, wenn $\mathfrak{F}$ ein Ultrafilter ist. Zu einem Ultrafilter $\mathfrak{A}\supseteq\mathfrak{F}$ gehört daher eine t-Bewertung von $\mathfrak{G}_\mathfrak{F}$ mit der Wertegruppe $\mathfrak{G}_\mathfrak{A}$. Verf. beweist, dass sich jede t-Bewertung von $\mathfrak{G}_\mathfrak{F}$ als Produkt einer zu einem Ultrafilter $\mathfrak{A}\supseteq\mathfrak{F}$ gehörigen t-Bewertung mit einer t-Bewertung von $\mathfrak{G}_\mathfrak{A}$ darstellen lässt. Als Anwendung ergibt sich, dass es für

Funktionen f, g genau dann ein $n>0$ mit $f\leqq ng$ mod $\mathfrak{H}_{\mathfrak{F}}$ gibt, wenn es für jeden Ultrafilter $\mathfrak{A}\supseteq\mathfrak{F}$ ein $n>0$ mit $f\leqq ng$ mod $\mathfrak{H}_{\mathfrak{A}}$ gibt. *P. Lorenzen* (Kiel).

Šik, František. Zur Theorie der halbgeordneten Gruppen. Czechoslovak Math. J. **6(81)** (1956), 1–25. (Russian. German summary)

The author considers the structure of lattice-ordered groups relative to the relation of disjointedness and a quasi-ordering $\prec$ ($a\prec b\Rightarrow|a|\leq|b|$). No assumptions are made about the completeness of the l-group G.

A component of G is the set of all elements of a l-group G that are disjoint from every element of a set $A\subset G$. It is shown that the system of components of a l-group G form a complete Boolean algebra Γ_0; the components that are l-ideals form a subalgebra Γ_1 and the direct factors of G a subalgebra Γ_2 of Γ_1. [See also Kantorovič, Vulih, and Pinsker, Functional analysis in partially ordered spaces, Gostehizdat, Moscow-Leningrad, 1950; Uspehi Mat. Nauk (N.S.) **6** (1951), no. 3(43), 31–98; MR **12**, 340; **13**, 361].

The direct factors of a l-group G receive considerable attention and the following is typical: A system $\{J_\alpha\}$ of factors J_α of an l-group G is called complete in G if $x\in G$, $x\,\delta\,\mathbf{U}_\alpha J_\alpha\Rightarrow x=0$. If $\{J_\alpha\}$ is a complete system of direct factors of a l-group G then G can be embedded in the complete direct sum $G^*=\sum_\alpha J_\alpha$ of the system $\{J_\alpha\}$.

Archemedian l-groups G and partially ordered l-groups are also considered. *L. J. Paige* (Los Angeles, Calif.).

Hewitt, Edwin; and Zuckerman, Herbert S. The l_1-algebra of a commutative semigroup. Trans. Amer. Math. Soc. **83** (1956), 70–97.

To quote from the authors' introduction: "The present paper may be described as an introduction to harmonic analysis on discrete commutative semigroups." In broad outline the paper may be described thus: §§ 2, 3, 7, 8, 9, 10 are dominated by the analogy with that theory of the group-algebra of a locally compact commutative group in which the Banach algebra structure is exploited; §§ 4, 5, 6 are more nearly purely algebraic; and § 11 is devoted to examples. One of the principal aims is to pair off algebraic properties of the semigroup G with those of its l_1-algebra $l_1(G)$. Topological properties of the "dual" semigroup $\hat{G}$ also enter the discussion. Only a few of the results obtained can be listed here.

The semigroups G considered are almost always assumed commutative. A semicharacter of G is a complex function χ on G such that χ is bounded, $\chi\not\equiv 0$, $\chi(xy)=\chi(x)\chi(y)$. $\hat{G}$ is the set of all such semicharacters. $l_1(G)$ may be identified with the set of all complex functions α on G for which $\|\alpha\|=\sum_{x\in G}|\alpha(x)|<+\infty$; it is made into a Banach algebra by defining convolution in an obvious way. It is shown (Theorem 2.7) that the non-trivial linear multiplicative forms on $l_1(G)$ are in one-one correspondence with the semicharacters χ, the form defined by χ being specified as

$$\alpha\to\sum_{x\in G}\alpha(x)\chi(x)=\hat{\alpha}(\chi).$$

Here $\hat{\alpha}$ may be thought of as the Fourier transform of α.

An identification of the radical of $l_1(G)$ is given, together with two criteria for the semisimplicity of $l_1(G)$. Semisimplicity obtains if either (i) $\hat{G}$ separates points of G (Theorem 3.5), or (ii) $x^2=y^2=xy$ implies $x=y$ (Theorem 5.8). In connection with the first criterion, the authors prove an interesting result (Theorem 3.4) which yields an analogue of the Weierstrass-Stone Theorem applicable to a space $l_\infty(X)$ with its weak topology.

A subset A of G is termed a set of relative units if to each $x\in G$ corresponds $a\in A$ such that $ax=xa=x$. Then (Theorems 7.3 and 7.5) $l_1(G)$ has a unit if and only if G admits a finite set of relative units.

On $\hat{G}$ one may introduce the least fine topology for which each $\hat{\alpha}$ is continuous; $\hat{G}$ is then locally compact (Hausdorff), each $\hat{\alpha}$ vanishes at infinity, and $\hat{G}$ is compact if $l_1(G)$ has a unit. Necessary and sufficient conditions for $\hat{G}$ to be compact are given in Theorems 8.7 and 8.12. Similar results are given in Theorems 9.1 and 10.6 concerning the existence of proper idempotents in $l_1(G)$ and the connectedness of $\hat{G}$ respectively.

The authors express their hope of dealing later with Tauberian theorems and analogues of the Pontryagin duality theory. *R. E. Edwards* (London).

Oganesyan, V. A. On the semisimplicity of a system algebra. Akad. Nauk Armyan. SSR. Dokl. **21** (1955), 145–147. (Russian. Armenian summary)

Let R be a finite semigroup in which all pairs of idempotents commute and in which the equation $axa=a$ has a solution x for every a. Theorem. The semigroup algebra of R over any field of characteristic zero is semisimple. This is a special case of a theorem proved independently by Munn [Proc. Cambridge Philos. Soc. **51** (1955), 1–15; MR **16**, 561]. The author also gives a finite semigroup whose semigroup algebra fails to be semisimple. [For a list of such semigroups, see Hewitt and Zuckerman, Acta Math. **93** (1955), 67–119; MR **17**, 1048.] *E. Hewitt.*

Munn, W. D. The characters of the symmetric inverse semigroup. Proc. Cambridge Philos. Soc. **53** (1957), 13–18.

This is a study of generalization $A^{(n)}$ of the symmetric group. It consists of all "partial transformations" of a set S of n objects. A partial transformation x is a one-to-one mapping of a subset A onto a subset B. The sets A and B have the same cardinal. The null set can only be mapped onto itself, and this transformation 0 is an element of $A^{(n)}$. An inverse of each partial transformation clearly exists, to which the word "inverse" in the title alludes. The product of two partial transformations x and x', which take A into B and A' into B' respectively, is a partial transformation $x\cdot x'$ from the set $x^{-1}(B\cap A')$ to the set $x(B\cap A')$. If $B\cap A'$ is the null set then $x\cdot x'=0$. The order of $A^{(n)}$ is $\sum_{r=0}^{n}\binom{n}{r}^2 r!$. The algebra of $A^{(n)}$ over a field (of characteristic prime or zero) is semisimple.

A method of computing the character values of the irreducible representations of $A^{(n)}$ is given, and illustrated for $A^{(4)}$ (which has order 209). *H. Campaigne.*

Vandiver, H. S.; and Weaver, M. W. A development of associative algebra and an algebraic theory of numbers. IV. Math. Mag. 30 (1956), 1–8.

The first two papers in this series were written by Vandiver [Math. Mag. **25** (1952), 233–250; **27** (1953), 1–18; MR **14**, 348; **15**, 202]. The present paper is a direct continuation of the third paper in the series, which was also written by both authors [ibid. **29** (1956), 135–151; MR **17**, 825]. In this fourth paper semi-rings are studied. They are first introduced as the set of endomorphisms of a semi-group (additive) and are then defined abstractly. The paper ends with a statement of a theorem that relates a homomorphism of a semi-ring to an ideal in that semi-ring. *H. W. Brinkmann* (Swarthmore, Pa.).

Climescu, Al. L'indépendance des conditions d'associativité. Bull. Inst. Politehn. Iaşi (N.S.) **1** (1955), 1–9. (Romanian. Russian and French summaries)

Given a set of n elements $a, b, \cdots$, the triple (x, y, z) is called isolable if it is possible to define a binary multiplication such that (x, y, z) is non-associative $(xy\cdot z \neq x\cdot yz)$ while every other triple of elements of the set is associative. G. Szász [Acta Sci. Math. Szeged **15**, 20–28 (1953), 130–142 (1954); MR **15**, 95, 773] showed that triples of the types (a, a, b), (b, a, a), (a, b, a), (a, b, c) are isolable if and only if $n\geqq 3$, while (a, a, a) is isolable if and only if $n\geqq 4$. In each case he gave an example of an algebra (i.e. of a multiplication table) of the minimal order n. The author improves on this by constructing all such algebras of minimal order. If isomorphic algebras and algebras with products reversed are not distinguished, there are just 3 such algebras having (a, a, b) [or, reversing products, (b, a, a)] isolated, and just one for each of (a, b, a) (a, b, c), (a, a, a).

The triple (x, y, z) is called isolable of the second kind if an algebra exists in which $xy\cdot z = x\cdot yz$ while every other triple of elements is non-associative. This is impossible if $n=2$, but every triple is isolable if $n\geqq 4$. ($n=3$ is not considered.)

These ideas are generalized. N-ary operations are considered, and isolation of either kind is defined not merely for a $(2N-1)$ple but for any set S of $(2N-1)$ples, or simultaneously for an ensemble of sets S_i relative to different finitary operations f_i. The general conclusion may be stated roughly: If the number of elements is sufficiently great, the conditions of associativity are independent, in the sense of isolability of either kind.

I. M. H. Etherington (Edinburgh).

See also: Adyan, p. 455; Climescu, p. 461; Zelinsky, p. 464; Nöbauer, p. 466; Jones and Watson, p. 467; Munn, p. 489; Helgason, p. 494; Lashof, p. 497; Pickert, p. 502.

THEORY OF NUMBERS

General Theory of Numbers

Chojnacka-Pniewska, M. M. Sur les congruences aux racines données. Ann. Polon. Math. **3** (1956), 9–12.

For every set of distinct residue classes $r_1, r_2, \cdots, r_k$ (mod m), can one construct a polynomial $f(x)$ with integer coefficients, such that $f(x)\equiv 0 \pmod m$ has precisely the roots $r_1, r_2, \cdots, r_k$? The author shows that the answer is yes when m is 4 or an odd prime, and no when $m=6, 8, 9, 10$.

{Reviewer's note: It is easy to prove that the answer is no for all composite $m>4$.} *L. Moser.*

Kužel', A. V. Elementary solution of Waring's problem for polynomials by the method of Yu. V. Linnik. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 165–168. (Russian)

Kamke [Math. Ann. **83** (1921), 85–112] gave the following generalization to polonymials of Hilbert's theorem on the Waring problem for nth powers: Let $f(x)$ be polynomial of degree n with integer coefficients the highest coefficient being positive, then there is a g depending on f such that every positive integer is the sum of not more than g numbers taken from the set

$$1, f(0), f(1), \cdots.$$

The present note gives an elementary demonstration of this theorem, using as a lemma a result Linnik [Mat. Sb. N.S. **12(54)** (1943), 225–230; MR **5**, 200] in his elementary proof of Hilbert's theorem by Schnirelmann's method of density. This lemma limits the number of solutions of the diophantine equation $\sum_{i=1}^{k} f(x_i)=m$ subject to the conditions $0\leqq x_i^n\leqq N$, $1\leqq m\leqq N$. *D. H. Lehmer.*

Palamà, Giuseppe. Su taluni problemi che si riducono a quello ideale di Escott-Tarry o di Prouhet-Tarry. Boll. Un. Mat. Ital. (3) **11** (1956), 569–577.

Rusu, Eugen. L'unicité de la décomposition en certains anneaux par des entiers quadratiques. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **8** (1956), 273–295. (Romanian. Russian and French summaries)

The uniqueness of the factorization into primes of integers in quadratic fields is studied without use of the Euclidean algorithm. Let D be squarefree and consider the form $f=x^2-Dy^2$. Let x and y run through all rational integers, so that $(x, y)=1$, and consider the set P of all rational prime divisors of the f's previously obtained. Uniqueness of factorization exists in $R(\sqrt{D})$ if and only if for every $p \in P$ one of the two relations $\pm p=a^2-Db^2$ holds with $(a, b)=1$, provided that $D\equiv 2$ or $3 \pmod 4$, or $D\equiv 1 \pmod 8$. If $D\equiv 5 \pmod 8$ the necessary and sufficient condition for the uniqueness of factorization is that at least one of the four equalities $\pm p=a^2-Db^2$, $\pm 4p=a^2-Db^2$, $(a, b)=1$ should hold for all $p \in P$. In particular, it follows that one cannot have uniqueness of factorization if $0<D\equiv 3 \pmod 4$ and D is composite, if $0<D\equiv 2 \pmod 4$ and D contains at least two odd primes, or if $0>D\equiv 3 \pmod 4$ and $|D|\geqq 5$. The proofs are elementary.

E. Grosswald (Philadelphia, Pa.).

Nöbauer, Wilfried. $\mathfrak{M}$-Untergruppen von Restklassengruppen. Monatsh. Math. **60** (1956), 269–287.

The present paper is to a great extent a continuation of the writer's earlier paper [Monatsh. Math. **59** (1955), 118–145; MR **15**, 906]. The concept of natural subgroups of the previous paper (q.v. for notation) is now replaced by that of $\mathfrak{M}$-subgroups. Let $k\geqq 1$. For each integer n let $\mathfrak{M}(n)$ denote a set of polynomial vectors of Γ_k. Let $\mathfrak{B}_n^k$ denote the set of all elements of $\Gamma_k|A_k^n$ that have a representative in $\mathfrak{M}(n)$. Put $\mathfrak{U}_n^k=\mathfrak{B}_n^k\cap\mathfrak{G}_n^k$; thus with each group $\mathfrak{G}_n^k$ is associated a complex $\mathfrak{U}_n^k\subset\mathfrak{G}_n^k$. Let $\{n\}$ denote the set of n for which $\mathfrak{U}_n^k$ is a subgroup of $\mathfrak{G}_n^k$; $\{n\}$ is called the domain of definition of the $\mathfrak{M}$-subgroup $\mathfrak{U}_n^k$. It is assumed that the following property V1 holds: The domain of definition of $\mathfrak{U}_n^k$ consists of all the natural numbers. Next it is shown that the property V2: if $m|n$ then $\mathfrak{M}(n)\subseteq\mathfrak{M}(m)$ implies the following result. If $n=ab$ and $(a, b)=1$, then $\mathfrak{U}_n^k$ is a subgroup of the direct product $\mathfrak{U}_a^k\times\mathfrak{U}_b^k$. In particular, if also $\mathfrak{U}_n^k=\mathfrak{U}_a^k\times\mathfrak{U}_b^k$, $\mathfrak{U}_n^k$ is called multiplicative; conditions are found for which this property holds. It is also proved that if V2 holds, then $m|n$ implies that $\mathfrak{U}_m^k$ is a homomorphic image of $\mathfrak{U}_n^k$. There are a number of additional theorems in this part of the paper making use of a third property V3. In the next place the writer allows k as well as n to vary. Thus he considers sets of polynomial vectors $\mathfrak{M}(n, k)$ and introduces three properties V1, V4, V5.

In the remainder of the paper the writer discusses in considerable detail five special examples of $\mathfrak{M}$-subgroups.

L. Carlitz (Durham, N.C.).

Schwarz, Štefan. On a type of universal forms in discretely valued fields. Acta Sci. Math. Szeged 17 (1956), 5–29.

In a previous paper [Quart. J. Math. Oxford Ser. **19** (1948), 123–128; MR 9, 572] the writer proved Theorem 1. Let $\delta=(p^f-1, k)\leq p-1$, and suppose that $a_1, \cdots, a_s$ are elements of $GF(p^f)$, $a_1a_2\cdots a_s\neq 0$. Then the equation $a_1x_1^k+\cdots+a_sx_s^k=b$ is solvable in $GF(p^f)$ for every $b\in GF(p^f)$. The object of the present paper is to extend this theorem in various ways. It is pointed out, by means of a special example, that Th. 1 is not a consequence of Chevalley's theorem.

Let K denote a field complete with respect to a discrete non-archimedian valuation. Let I be the ring of integers $\in K$, π a prime of K, $\mathfrak{p}=\pi I$ the prime ideal generated by π; it is assumed that the residue class field $I=I/\mathfrak{p}$ is a finite field $GF(p^f)$. Two cases are considered. (A) K has characteristic 0; it is the derived field of an algebraic number field $R(\theta)$ complete under a valuation corresponding to a prime ideal $\mathfrak{p}$ of $J[\theta]$, the ring of algebraic integers $\in R(\theta)$. (B) K has characteristic p and is the field of formal power series in x, containing only a finite number of negative powers, with coefficients $\in GF(p^p)$. In either case, every $a\in K$ is representable in the form

$$a=\pi^{-\nu}(a_0+a_1\pi+\cdots)\ (a_0\neq 0,\ a_i\in\Re).$$

In case (A) we choose $\pi\in J[\theta]$ such that $\mathfrak{p}/\pi$, $\mathfrak{p}^2\nmid\pi$; $\Re$ denotes a complete residue system (mod $\mathfrak{p}$). The principal results of the paper are the following.

Theorem 2. Let K be a complete field of type (A) or (B). Assume that (a) $k>1$, $(k, p)=1$, (b) $\delta=(k, p^f-1)\leq p-1$, (c) $a_1, \cdots, a_{s+1}\in K$, $a_1a_2\cdots a_{s+1}\not\equiv 0$ (mod $\mathfrak{p}$). Then the equation

$$a_1x_1^k+\cdots+a_{s+1}x_{s+1}^k=a$$

is solvable in K for every $a\in K$. Theorem 2a. Assume now that $a_1a_2\cdots a_s\not\equiv 0$ (mod $\mathfrak{p}$). Then if the congruence

$$a_1x_1^k+\cdots+a_sx_s^k\equiv 0 \pmod{\mathfrak{p}}$$

has at least one non-zero solution, then every $b\in K$ can be written in the form

$$b=a_1x_1^k+\cdots+a_sx_s^k\ (x_i\in K).$$

Put $k=k_0p^t$, $p\nmid k_0$, $t\geq 0$. If $t\geq 1$, then it is shown that in case (B) the equation

$$a=a_1x_1^k+\cdots+a_sx_s^k$$

need not be solvable in K for any $s\geq 1$. For case (A) we have Theorem 3. Let $\delta=(k, p^f-1)\leq p-1$,

$$s=(p^{f(t+1)}-1)\delta/(p^f-1),$$

and suppose that $a_1, \cdots, a_s$ are s element of $J[\theta]$ such that $a_1a_2\cdots a_s\not\equiv 0$ (mod $\mathfrak{p}$). Then the congruence

$$b\equiv a_1x_1^k+\cdots+a_sx_s^k \pmod{\mathfrak{p}^{t+1}}$$

is solvable for every $b\in J[\theta]$. Theorem 4. With the same notation as in the previous theorem and $\mathfrak{p}\mid p$, $\mathfrak{p}^2\nmid p$, the equation

$$b=a_1x_1^k+\cdots+a_{s+1}x_{s+1}^k$$

is solvable in K for every $b\in K$. Theorem 4a. Suppose now that $a_1a_2\cdots a_s\not\equiv 0$ (mod $\mathfrak{p}$). Then if the congruence

$$0\equiv a_1x_1^k+\cdots+a_sx_s^k \pmod{\mathfrak{p}^{t+1}}$$

has at least one non-zero solution, it follows that the equation

$$b=a_1x_1^k+\cdots+a_sx_s^k$$

is solvable in K.

{Reviewer's remark. In connection with Th. 3 reference may be made to a recent paper by Eckford Cohen, Congruences in algebraic number fields, Transactions of the American Math. Society, **83** (1956), 547–556.}

L. Carlitz (Durham, N.C.).

Linnik, Yu. V. The asymptotic geometry of the Gaussian genera; an analogue of the ergodic theorem. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 1018–1021. (Russian)

In a previous paper [Vestnik Leningrad. Univ. **10** (1955), no. 2, 3–23, no. 5, 3–32, no. 8, 15–27; MR **18**, 193] the author obtained results on the distribution, as $D\to\infty$, of the integers (a, b, c) which correspond to reduced positive definite forms $ax^2+2bxy+cy^2$ of odd determinant $-D$, $D=ac-b^2$. The present paper indicates how the method can be developed to give similar results concerning the forms in a particular genus (in the sense of Gauss). As previously, the result is imperfect to the extent that it involves a prime p for which $(-D/p)=1$, and consequently the uniformity of distribution (in the sense of Lobatchevsky area) follows at once only if $D\to\infty$ through values for which p remains bounded. The main result is as follows. Represent each form by the point $x_1=aD^{-1/2}$, $x_2=cD^{-1/2}$, $x_3=bD^{-1/2}$, lying on that portion, say Δ_0, of the hyperbolic surface $x_1x_2-x_3^2=1$ which is cut off by the inequalities defining reduction. Let Σ_0 be a simply connected region contained in Δ_0 and having a piece-wise smooth boundary. Let R be any one of the genera, the numbers of which is $2^{\beta(D)}$. Let $H_R(\Sigma_0, D)$ denote the the number of reduced forms of determinant $-D$ in the genus R with representative points in Σ_0. Then

$$H_R(\Sigma_0, D)=\frac{9}{2\pi}A(\Sigma_0)2^{-\beta(D)}h(-D)(1+\eta(p, D)),$$

where $A(\Sigma_0)$ is the area of Σ_0 in the sense of Lobatchevsky, $h(-D)$ is the total number of reduced forms of determinant $-D$, and $\eta(p, D)\to 0$ as $D\to\infty$ for fixed p. The detailed proof is not given, but it is indicated that it depends on an "ergodic theorem" to the effect that, for almost all forms $\varphi(x, y)$, the forms $\varphi^\nu(x, y)$ obtained by applying a particular linear transformation ν times, are uniformly distributed for a long sequence of ν, all this to be interpreted asymptotically as $D\to\infty$. *H. Davenport.*

Jones, B. W.; and Watson, G. L. On indefinite ternary quadratic forms. Canad. J. Math. **8** (1956), 592–608.

For an indefinite ternary quadratic form f with determinant d the author defines a multiplicative group Γ_d of square free integers prime to d. Further, he shows that Γ_d has a subgroup $\gamma(f)$ consisting of all those elements of Γ_d which are the l.c.m. of the denominators of the elements of the matrices of the rational automorphs of f and proves that the number of classes in the genus of f is equal to the order of the factor group $\Gamma_d/\gamma(f)$. Moreover, if n is represented by at least one but not all of the classes of forms in the genus of f, then the number of classes in the genus that represent n is equal to the number of classes that do not. *W. H. Simons.*

See also: Sprott, p. 459; Postnikov, p. 468; Heegner, p. 476.

Analytic Theory of Numbers

Bellman, Richard. A generalization of some integral identities due to Ingham and Siegel. Duke Math. J. **23** (1956), 571–577.

The main identity (corrected) is

$$(2\pi i)^{-\frac{1}{2}p(p+1)}\int\cdots\int e^{\operatorname{tr}(CS)}|S_p|^{-k_p}|S_{p-1}|^{-k_{p-1}}\cdots|S_1|^{-k_1}\prod_{i\le j}ds_{ij}$$

$$(*)\qquad =\frac{(2\sqrt{\pi})^{-\frac{1}{2}p(p-1)}|C^{(1)}|^{\sum k_i-\frac{1}{2}(p+1)}|C^{(2)}|^{-k_1}\cdots|C^{(p)}|^{-k_{p-1}}}{\Gamma(k_p)\Gamma(k_p+k_{p-1}-\frac{1}{2})\cdots\Gamma(\sum k_i-\frac{1}{2}p+\frac{1}{2})}$$

if C is positive definite, and $=0$ otherwise. Here the summations run from 1 to p, $C=(c_{ij})$ and $S=(s_{ij})$ are symmetric $p\times p$ matrices (C fixed and real, S variable and complex), $S_k=(s_{ij})$ $(1\le i,j\le k)$, $C^{(k)}=(c_{ij})$ $(k\le i,j\le p)$; $|X|$ denotes the determinant of the matrix X; the $\frac{1}{2}p(p+1)$ integrations are along lines on which $A=(a_{ij})$ is positive definite, where $a_{ij}=\Re(s_{ij})$; and the parameters k_i are real with k_p sufficiently large. Formally the argument follows the lines of the known proof for $k_1=\cdots=k_{p-1}=0$ [Ingham, Proc. Cambridge Philos. Soc. **29** (1933), 271–276], but omits discussion of logical points such as convergence. {The author's statement seems to imply that it is sufficient for convergence (in some sense) that the arguments of all Γ-functions be positive, but this is not so. Thus, if $p=1$, $C=0$, $A=I$, the integral is divergent when $0<k_1<1$.} The Laplace inverse of (*) is stated. The identities are used to generalize Siegel's matrix analogue of an identity of Lipschitz connected with the functional equation of the Riemann zeta-function, and to determine the result of applying the operator $|\partial/\partial c_{ij}|$ $(i,j=1,\cdots,K)$ to power-products $|C^{(1)}|^{a_1}\cdots|C^{(p)}|^{a_p}$. There are numerous inaccuracies of detail.

A. E. Ingham (Cambridge, England).

Levinson, Norman. On closure problems and the zeros of the Riemann zeta function. Proc. Amer. Math. Soc. **7** (1956), 838–845.

This paper is concerned with conditions that the Riemann zeta-function $\zeta(s)$ should have no zeros in certain regions of the s-plane. The main result is that if λ_n is a positive increasing sequence such that $\sum\lambda_n^{-1}$ is divergent, a necessary and sufficient condition that $\zeta(s)$ should have no zeros in the strip $\sigma_1<\operatorname{re}\ s<\sigma_2$, where $\frac{1}{2}\le\sigma_1<\sigma_2\le 1$, is that given any $\varepsilon>0$ and α and β such that $\sigma_1<\alpha<\beta<\sigma_2$, there exists an integer N and numbers a_n, $n=1,\cdots N$, (depending on ε, α and β) such that

$$\int_0^\infty\Big(\sum_1^N a_n\frac{e^{-\lambda_n x}}{1+e^{-\lambda_n x}}-e^{-x}\Big)^2\Big(x^{2\alpha-1}+x^{2\beta-1}\Big)dx<\varepsilon.$$

E. C. Titchmarsh (Oxford).

Postnikov, A. G. Additive problems with a growing number of terms. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 392. (Russian)

Asymptotic expressions are stated for the number of solutions of the diophantine equations

$$(1)\qquad x_1+x_2+\cdots+x_n=N,$$

$$(2)\qquad x_1^2+x_2^2+\cdots+x_n^2=N,$$

obtained by considering the addition of independent random variables, applying the local limit theorem of probability theory and estimating certain resulting trigonometric sums. *W. H. Simons* (Vancouver, B.C.).

Negoescu, Nicolae. Approximation asymétrique des nombres irrationnels par des rationnels. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **1** (1955), 21–30. (Romanian. Russian and French summaries)

Let θ be an irrational number having the regular continued fraction expansion $[a_0, a_1, \cdots, a_n, \cdots]$ and, for any integer r set

$$\lambda_r=[a_{r+1}, a_{r+2}, \cdots]+[0, a_r, a_{r-1}, \cdots, a_1].$$

Given $\tau>0$, let μ be such that the inequality

$$-1/\mu q^2<p/q-\theta<\tau/\mu q^2$$

is satisfied by infinitely many convergents p/q of θ. If $M(\theta,\tau)=\limsup\mu$, then the author has shown [C. R. Acad. Sc. Paris **226** (1948), 1495–1497; MR **9**, 569] that $M(\theta,\tau)=\max[\limsup\lambda_{2\nu},\ \tau\limsup\lambda_{2\nu+1}]$. For every given irrational θ, $M=M(\theta)$ represents a broken line with vertex P at $\tau=\limsup\lambda_{2\nu}/\limsup\lambda_{2\nu+1}$, $M=\limsup\lambda_{2\nu}$ and the set $\mathfrak{M}$ of all such points P has the power of the continuum. Properties of $\mathfrak{M}$ and of some of its subsets, corresponding to quadratic irrationalities are studied.

E. Grosswald (Philadelphia, Pa.).

Negoescu, Nicolae. Une méthode arithmétique pour le problème des approximations asymétriques. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **1** (1955), 31–38. (Romanian. Russian and French summaries)

Let θ be an irrational number and denote by $[a_0, a_1, \cdots, a_n, \cdots]$ its expansion in a regular continued fraction. Denote the denominator of the convergent $[a_{n+1}, a_{n+2}, \cdots, a_m]$ by $r_{n,m}$ and set $\varepsilon_n=\frac{1}{2}(1-(-1)^n)$. Then, for integers $n<m<l$, $m-n\equiv l-m\equiv 1 \pmod 2$ and arbitrary $\tau>0$, at least one of the three convergents p_n/q_n, p_m/q_m, p_l/q_l of θ satisfies $-1/\xi q^2<p/q-\theta<\tau/\xi q^2$ with

$$\xi=\Big[\Big(\frac{\tau^{\varepsilon_l}r_{n,m}^2+\tau^{\varepsilon_n}r_{m,l}^2+\tau^{\varepsilon_m}r_{n,l}^2}{r_{n,m}r_{m,l}r_{n,l}}\Big)^2-4\frac{\tau^{\varepsilon_n+\varepsilon_l}}{r_{n,l}^2}\Big]^{1/2}.$$

This theorem generalizes a result by Fujiwara [Proc. Imp. Acad. Tokyo **2**, (1926), 1–3] and contains as particular cases theorems of Vahlen, Hurwitz, Borel, Khintchine, Humbert and Segre on the (symmetric and asymmetric) approximations of irrational numbers by rationals. The method used is an extension of that of Fujiwara.

E. Grosswald (Philadelphia, Pa.).

See also: Schwarz, p. 467; Erdös et Karamata, p. 478.

Theory of Algebraic Numbers

★ **Yamazaki, Keijiro. Fibre spaces and sheaves in number theory.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 93–101. Science Council of Japan, Tokyo, 1956.

Pursuing the analogy between algebraic number theory and algebraic geometry, the author proposes here to introduce the notion of varieties, sheaves, etc. into algebraic number theory. The main idea is as follows.

Let Q be the rational field, Z the ring of rational integers and $Z_p=Z/(p)$. Let K and K_p be universal domains over Q and Z_p, respectively, in the sense of algebraic geometry, and let R be a subring of K containing Z such that, for each p, K_p is the algebraic closure of the image of a homomorphism φ_p of R into K_p. For any polynomial $f(X)=f(X_1,\cdots,X_n)$ in $R[X]$, the polynomial in $K_p[X]$ obtained by replacing the coef-

ficients a_i of $f(X)$ by $\varphi_p(a_i)$ will be denoted by $f_p(X)$. Let S^n be the set-theoretical union of the n-spaces $K_p{}^n$ over K_p for all prime numbers p. Every polynomial $f(X_1, \cdots, X_n)$ in $R[X]$ then gives a function $f(\xi)$ on S^n with values in S^1, defined by $f(\xi)=f_p(\xi_1, \cdots, \xi_n)$ when $\xi=(\xi_1, \cdots, \xi_n)$ is in $K_p{}^n$. Using such functions, a sheaf structure can be defined on S^n just as in algebraic geometry and "affine varieties" in S^n are defined as well.

Now, let F be a finite algebraic number field and $\omega_1, \cdots, \omega_n$ elements in F such that $Z[\omega_1, \cdots, \omega_n]$ is the ring of all algebraic integers in F. An affine variety V in S^n is then defined by the set of all ξ in S^n satisfying $f(\xi)=0$ for every polynomial $f(X)$ in $Z[X]$ such that

$$f(\omega_1, \cdots, \omega_n)=0,$$

and it is noticed that the field of rational functions on V defined over Z is isomorphic with F and that many arithmetic properties of F may be given geometric interpretations using the variety V. *K. Iwasawa.*

★**Ono, Takashi. On orthogonal groups over number fields.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 253–256. Science Council of Japan, Tokyo, 1956.

Soient V, W deux espaces vectoriels de dimensions n, m sur un corps commutatif K; soit f (resp. g) une forme bilinéaire symétrique non dégénérée sur V (resp. W) et soient $O(V, f)$, $O(W, g)$ les groupes orthogonaux correspondants. L'auteur étudie le problème suivant (pour $n>m$): existe-t-il une application semi-linéaire u de W dans V (relative à un automorphisme σ de K) telle que g soit proportionnelle à la forme bilinéaire

$$(x, y)\to(f(u(x), u(y)))^{\sigma-1}?$$

Utilisant la théorie classique de Hasse, il indique des conditions nécessaires et suffisantes pour qu'il existe une telle application lorsque K est un corps localement compact de caractéristique $\neq 2$, ou lorsque K est un corps de nombres algébriques ou un corps de fonctions algébriques d'une variable sur un corps fini de caractéristique $\neq 2$. Il remarque aussi que le problème précédent équivaut à la possibilité de plonger $O(W, g)$ dans $O(V, f)$. La note ne contient pas de démonstrations. *J. Dieudonné.*

See also: Inaba, p. 464; Rusu, p. 466; Schwarz, p. 467; Linnik, p. 467; Azumaya, p. 470.

ANALYSIS

Functions of Real Variables

Klement'ev, Z. I. Condition for compactness of a family of continuous functions. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 13–14. (Russian)

The author proves the following criterion for relative compactness of a set X of continuous functions defined on a closed interval $[a, b]$: For each sequence $\{x_n(t)\}$ chosen from X the functions $\sup_n x_n$, $\limsup_n x_n$, $\inf_n x_n$, and $\liminf_n x_n$ are all continuous. *M. M. Day.*

Seidenberg, A. Some remarks on Hilbert's Nullstellensatz. Arch. Math. **7** (1956), 235–240.

Careful discussion of some constructive aspects of Hilbert's Nullstellensatz. *G. Kreisel.*

See also: Pickert, p. 456; Arsove, p. 472; Višik, p. 487; Inaba, p. 493.

Measure, Integration

Stolyarov, N. A. A generalization of the Stieltjes integral. Ukrain. Mat. Ž. **8** (1956), 330–334. (Russian)

The writer defines and gives some properties of a second order Stieltjes integral studied in the case $\psi(x)=x$ by Hahn [Akad. Wiss. Wien. Math.-Nat. Kl. S.B. IIa. 134 (1925), 449–470]. For $a=x_0<x_1<\cdots x_n=b$, let $\Delta\psi_k=\psi(x_{k+1})-\psi(x_k)$; then the integral

$$\int_b^a f(x)\frac{d^2\varphi(x)}{d\psi(x)}$$

is the limit as $\max \Delta x_k\to 0$ of

$$\sum_{k=1}^{n-1}\frac{\Delta\psi_k\varphi(x_{k-1})-(\Delta\psi_k+\Delta\psi_{k-1})\varphi(x_k)+\Delta\psi_{k-1}\varphi(x_{k+1})}{\Delta\psi_k\Delta\psi_{k-1}}.$$

Some conditions sufficient for existence of this integral are given; one integration by parts formula connects this with the Hellinger integral. *M. M. Day.*

Neugebauer, Christoph J. A characterization of the Lebesgue area. Amer. J. Math. **79** (1957), 73–79.

Let $\mathfrak{T}$ be the class of all continuous mappings (T, A) from an admissible set A into E_3 and let $\mathfrak{F}$ be the collection of all functionals Φ defined on $\mathfrak{T}$ which are non-negative, lower semi-continuous, satisfy Kolmogoroff's principle, are monotone, and give the elementary area for all elementary mappings. [For convergence in $\mathfrak{T}$ and general terminology, see L. Cesari, Surface area, Princeton, 1956; MR **17**, 596.] The Lebesgue area L belongs to $\mathfrak{F}$.

A mapping $(T, A)\in\mathfrak{T}$ satisfies condition (s) provided that for each $\varepsilon>0$ there exists an elementary mapping (T', R) with $R\subset A$ and

(1) $$|T'(w)-T'(w')|\leq|T(w)-T(w')|,\ w, w'\in R,$$

(2) $$L(T, A)-L(T', R)<\varepsilon \text{ if } L(T, A)<+\infty,$$
$$L(T', R)>1/\varepsilon \text{ if } L(T, A)=+\infty.$$

Define

$$\mathfrak{T}(\Phi)=\{(T, A)\in\mathfrak{T}|\Phi(\mathfrak{T}, A)=L(T, A)\},$$
$$\mathfrak{T}^*=\bigcap\mathfrak{T}(\Phi),\ \Phi\in\mathfrak{F},$$
$$\mathfrak{T}'=\{(T, A)\in\mathfrak{T}|\ (T, A) \text{ satisfies condition } (s)\}.$$

The author shows that $\mathfrak{T}^*=\mathfrak{T}'$. Thus condition (s) characterizes those mappings for which all elements of F agree (with L). *E. Silverman* (E. Lansing, Mich.).

Džrbašyan, M. M.; and Tamadyan, A. P. On best approximation by entire functions in the complex domain. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 485–512. (Russian)

The present paper gives the proofs to the eight theorems which the authors published in Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 345–348 [MR **17**, 598] and which connect the best approximation of a function $f(z)$ on a closed

set $\bar{D}_\rho$ (where $\rho \geqq 1$), consisting of the union of two opposite angles with aperture $(1-\rho^{-1})\pi$, by entire functions of order ρ and of type $\leqq\sigma$ with the differential properties of (fz) on $\bar{D}_\rho$. *U. W. Hochstrasser.*

Leader, Solomon. Convergence topologies for measures and the existence of transition probabilities. Pacific J. Math. **6** (1956), 479–490.

This is mainly a technical study of several different topologies on various spaces of set-functions and the determination of their dual spaces. A useful byproduct is the observation that a stochastic semigroup (i.e., a one-parameter semigroup of norm-preserving positive linear operators on a space of measures) arises from a system of stationary transition probabilities on the underlying space if and only if it (the semigroup) is continuous in one of the topologies discussed earlier (the topology of bounded convergence). *P. R. Halmos* (Chicago, Ill.).

★ **Azumaya, Goro. An existence theorem of algebras.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 233–235. Science Council of Japan, Tokyo, 1956.

Let R be an algebra over a complete local ring K such that the semi-simple residue class algebra of R modulo its Jacobson radical N is separable over the residue class field of K. It is shown that there exists an unramified subalgebra S such that every residue class of R modulo N is represented by an element of S. Furthermore S is uniquely determined up to inner automorphisms. This result simultaneously generalizes two well-known theorems: the arithmetic theorem on the existence of the field of inertia, and the algebraic Wedderburn-Malcev theorem.
W. H. Mills (New Haven, Conn.).

See also: Bagemihl, p. 456; Standish, p. 479; Fleming, p. 489; Rudin, p. 489; Ionescu Tulcea, p. 490; Luxemburg, p. 491; Rosen, p. 495.

Functions of Complex Variables

Künzi, H. P. Einführung in die Theorie der quasikonformen Abbildungen. Elem. Math. **11** (1956), 121–129.

Expository article on the elementary theory of quasiconformal mappings. *L. Ahlfors* (Cambridge, Mass.).

Bers, Lipman. An outline of the theory of pseudoanalytic functions. Bull. Amer. Math. Soc. **62** (1956), 291–331.

Cet article constitue un exposé synthétique, extrêmement clair, de la théorie des fonctions pseudo-analytiques (en abrégé, p.a.) dans son état actuel; et il rassemble fort utilement tous les résultats (en grande partie dûs à l'A.) obtenus dans cette voie, ainsi que les problèmes en connexion avec cette théorie. La notion de fonction p.a. (dûe à l'A.) est définie ici de façon dogmatique; les générateurs F, G, étant 2 fonctions complexes continues dans un domaine plan D, ou, plus généralement, sur une surface de Riemann, satisfaisant à $\Im(\bar{F}G)>0$, la classe $A_D(F,G)$ des fonctions (F,G) p.a. de 1ère espèce est la classe maximale des fonctions $w=\varphi F+\psi G$ telles que si $w(z_0)=0$, $w(z)/(z-z_0)$ ait une limite en z_0; et à w on fait correspondre la fonction $\omega=\varphi+i\psi$, dite p.a. de 2ème espèce. La condition imposée à w équivaut à $\varphi_{\bar{z}}F+\psi_{\bar{z}}G=0$, et la (F,G) dérivée de w est $\dot{w}=\varphi_z F+\psi_z G$. F et G seront d'abord supposées satisfaire seulement à une condition de Hölder, ce qui entraîne la continuité de $\dot{w}$. La (F,G) dérivée [resp. la (F,G) primitive] de w est p.a. pour un couple de générateurs (F_1,G_1) [resp. (F_{-1},G_{-1})] dit successeur [resp. prédécesseur] de (F,G).

Les propriétés classiques des fonctions analytiques sont passées en revue, et étendues aux fonctions p.a.: les types de singularités sont les mêmes; les fonctions p.a. de 2ème espèce définissent des transformations intérieures; les puissances successives de z sont remplacées par une suite de fonctions Z_ν, définies par récurrence, qui permettent le développement formel en série de Taylor d'une fonction p.a. de 1ère espèce w. La partie II est consacrée aux propriétés globales des fonctions p.a.: tout d'abord, si F et G sont définies dans le plan compactifié (sphère de Riemann), on définit des séries formelles d'exposants positifs et négatifs, qui permettent d'obtenir une formule intégrale de Cauchy; les exposants fractionnaires conduisent à des fonctions p.a. multiformes; si F et G sont doublement périodiques, on peut définir des fonctions p.a. „elliptiques". De même, la théorie de l'uniformisation, et celles des différentielles définies sur une surface de Riemann, avec le théorème de Riemann-Roch, s'étendent aux fonctions p.a. Dans la partie III les générateurs F, G, sont supposés différentiables. Au couple (F,G) sont alors associés des „coefficients caractéristiques" a, b, A, B, tels que la pseudo-analycité de 1ère espèce soit caractérisée par l'une des propriétés suivantes: 1) $w_{\bar{z}}=aw+b\bar{w}$, 2) la fonction

$$h(z)=w(z)+\frac{1}{\pi}\iint_D \frac{a(\zeta)w(\zeta)+b(\zeta)\overline{w(\zeta)}}{\zeta-z}\,d\xi d\mu$$

est analytique. Sous certaines conditions, w étant p.a. dans un domaine D, il existe une fonction analytique $f(z)$ et une fonction $s(z)$ telles que $w=e^{s(z)}f(z)$. Ce principe de similitude permet d'étudier la convergence des séries de Taylor formelles précédemment définies. L'A. étudie ensuite les relations des fonctions p.a. avec les solutions d'équations linéaires elliptiques à 2 variables, et même avec celles d'une équation de type mixte. Pour terminer, il indique diverses généralisations, et certains résultats récents de la théorie des équations elliptiques, linéaires ou quasi-linéaires. Une abondante bibliographie termine cet exposé. *J. Lelong* (Paris).

Mişicu, M. Résolution de certains problèmes d'équilibre des milieux continus dans l'espace. I. Procédés de calcul à l'aide de fonctions quaternioniques. Com. Acad. R. P. Romîne **6** (1956), 71–82. (Romanian. Russian and French summaries)

In the present paper the author introduces the formal method of quaternions for studying the rotational motion of perfect incompressible fluids, vortices, equilibrium of elastic bodies, etc. The analysis is similar to that of complex-variable theory [cf. G. Moisil, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **14** (1931), 401–408]. *K. Bhagwandin* (Oslo).

Mişicu, M. Résolution de certains problèmes d'équilibre des milieux continus dans l'espace. II. Représentation du mouvement des fluides parfaits incompressibles. Com. Acad. R. P. Romîne **6** (1956), 83–87. (Romanian. Russian and French summaries)

The author applies the analysis of the paper reviewed above to the study of the rotational motion of perfect incompressible fluids. {As far as the present reviewer can ascertain, the author's theory does not yield the necessary explicit solution to the problem in question. Also, the

author does not seem to be aware of the appropriate boundary-value problems [cf. Fischer, Universal mechanics and Hamilton's quaternions, Axion Inst., Stockholm; MR 13, 502].} *K. Bhagwandin* (Oslo).

Seleznev, A. On functions monogenic on nowhere closed sets and sets of the type F_σ. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 591–594. (Russian)

L'auteur étend la théorie des fonctions monogènes de Borel aux ensembles du type F_σ, plus généraux que les „domaines de Cauchy" introduits par le créateur de ces fonctions. *S. Mandelbrojt* (Paris).

Páter, Z. On functions of a generalized complex variable. Gaz. Mat. Fiz. Ser. A. **8** (1956), 399–405. (Romanian. Russian and French summaries)

Haimo, Deborah Tepper. A note on convex mappings. Proc. Amer. Math. Soc. **7** (1956), 423–428.

A recent theorem of R. F. Gabriel [same Proc. **6** (1955), 58–66; MR **16**, 807] is generalised as follows. If

$$f(z)=z^{-1}+a_0+a_1z+a_2z^2+\cdots$$

and $|\{f(z), z\}|\leqq 2q(|z|)$ for $|z|<1$ then $f(z)$ maps $|z|<1$ onto the complement of a convex domain. (The differential equation $y''+q(y)y=0$ is to have a solution $y(x)$ which does not vanish in $0<x<1$ and satisfies the boundary conditions $y(0)=0$, $\lim_{x\to 1} y'(x)/y(x)\geqq\frac{1}{2}\cdot\{f(z), z\}$ is the Schwarzian derivative.) *A. J. Macintyre* (Aberdeen).

Delange, H.; und Gaier, D. Über asymptotische Wege analytischer Funktionen und ihrer Ableitungen. Arch. Math. **7** (1956), 135–142.

Let $f(z)$ be regular in G which is in $\Re(z)>0$ and contains the positive real axis $z=x>0$. Then if $f(z)=O\{\psi(r)\exp\varphi(r)\}$ with $r=|z|$, $z\in G$ and $f(x)=O\{\psi(x)\}$ for $x>0$ the authors show that $f(z)=O\{\psi(r)\}$ in a certain part of G containing $z=x>0$ and that $f^{(p)}(x)=O[\psi(x)\{\varphi(x)/\delta(x)\}^p]$. All O's relate to $|z|\to\infty$, $\delta(x)$ is the distance from $z=x>0$ to the boundary of G. The relations $\varphi'(x)/\varphi(x)=O\{1/\delta(x)\}$ and $\psi'(x)/\psi(x)=O\{1/\delta(x)\}$ are assumed.

It is also proved for example that $f'(z)$ will tend to zero if $f(z)$ does so, as z tends to infinity along a continuous curve, provided $f(z)$ is an integral function of finite type, order $\frac{1}{2}$. [Cf. D. Gaier, Bull. Amer. Math. Soc. **61** (1955), 93.] *A. J. Macintyre* (Aberdeen).

Rubel, L. A. Uniqueness properties of sets with blocks. Proc. Amer. Math. Soc. **7** (1956), 1023–1026.

Rubel, L. A. Necessary and sufficient conditions for Carlson's theorem on entire functions. Trans. Amer. Math. Soc. **83** (1956), 417–429.

Let $f(z)$ be an entire function of exponential type τ, such that $\limsup |y|^{-1}\log|f(iy)|=c$. Carlson's theorem states that if $c<\pi$ and $f(n)=0$ for all positive integers n then $f(z)\equiv 0$. These papers are concerned with the question of when the same conclusion can be drawn if $f(n)=0$ only for integers belonging to some set A. The first paper shows that for a given τ and a given c it is sufficient that A has blocks of consecutive integers $n_k<n<\phi n_k$, where $\phi>\phi(c,\tau)$; but this is not true if f is merely regular and of exponential type for $x>0$, even if the blocks are $n_k<n<\phi(k)n_k$ with $\phi(k)\to\infty$. The second paper obtains more precise results for a less sharply defined class of functions: in order that $f(z)\equiv 0$ follows from $f(n)=0$, $n\in A$, provided merely that $\tau<\infty$ and $c<\pi$, it is necessary and sufficient that $\Delta(A)=1$, where $\Delta(A)$ is the upper density, that is, $\limsup t^{-1}A(t)$, where $A(t)$ is the counting function of A. The surprising part of this result is that $\Delta(A)=1$ is sufficient to ensure that $f(z)\equiv 0$. The author proves it by showing first that it is true with $\Delta(A)$ replaced by the logarithmic block density

$$L(A)=\lim_{\lambda\to\infty}\limsup_{t\to\infty}(\lambda t)^{-1}\int_1^{\lambda t} v^{-1}dA(tv).$$

and then that $\Delta(A)=1$ if and only if $L(A)=1$. The question then arises of whether the main theorem can be extended to sets Λ of positive numbers λ_n satisfying the spacing condition $\lambda_{n+1}-\lambda_n\geqq\gamma>0$; the author conjectures that in order for $f(z)\equiv 0$ to follow from $f(\lambda_n)=0$ provided that $\tau<\infty$ and $c<C$ it is necessary and sufficient that $L(\Lambda)\geqq C/\pi$. He proves the sufficiency, and remarks that the necessity cannot be established by the device, available for sets of integers when $C=\pi$, of using $\prod(1-z^2/\lambda_n^2)$ as an example.

The paper contains a number of results about the relationships among various kinds of upper densities for sets $\{\lambda_n\}$. These are upper density Δ as defined above; upper mean density $\Delta_M=\limsup t^{-1}\int_0^t x^{-1}\Lambda(x)dx$; upper Abel density $\Delta_\alpha=\limsup y^{-1}\int_0^\infty e^{-t}d\Lambda(ty)$; upper Poisson density $\Delta_P=\pi^{-1}\limsup y^{-1}\int_0^\infty \log(1+t^{-2})d\Lambda(ty)$; and logarithmic block density $L(\Lambda)$ as defined above. It was known that $\Delta_P\leqq\Delta_M\leqq\Delta$ and $\Delta_\alpha\leqq\Delta_M$. The following additional results are obtained: $L(\Lambda)\leqq\Delta_P(\Lambda)\leqq\Delta_\alpha(\Lambda)$, with inequality possible in either place for sets of integers; if Λ consists of integers, then if any of Δ, Δ_M, Δ_α, Δ_P or L is 1, all of them are 1.

Finally the author shows that if $f(z)=\prod(1-z^2/\lambda_n^2)$ then $\limsup x^{-1}\log|f(x)|\geqq\pi(\Delta_M-\Delta_P)$. The interest of this result lies in the fact that if Λ has a density then $\limsup x^{-1}\log|f(x)|=0$, so that although the latter condition does not imply the existence of a density it does imply that the sequence $\{\lambda_n\}$ cannot be so irregular that $\Delta_M\neq\Delta_P$. *R. P. Boas, Jr.* (Evanston, Ill.).

Edrei, Albert. On a conjecture of Pólya concerning the zeros of successive derivatives. Scripta Math. **22** (1956), 31–44, 106–121.

Pólya's hypothetical theorem is that if $f(z)$ is a real entire function of order greater than 1 and is bounded on the real axis then the closure of the set of zeros of the successive derivatives of f contains the real axis. The author proves two theorems bearing on the problem and a number of other results from the same circle of ideas. (1) Let $f(z)$ be a real entire function of positive finite order ρ and mean type. Let $\limsup_{x\to+\infty} x^{-\rho}\log|f(\pm x)|<0$. Then $\rho>1$ and the set of real zeros of the successive derivatives of f is dense on the real axis. (2) We say that z_0 belongs to the final set of f if every neighborhood of z_0 contains zeros of infinitely many derivatives of f. Let f be a real transcendental entire function satisfying a linear differential equation with polynomial coefficients. Then for some positive finite ρ we have $\lim r^{-\rho}\log M(r)=\sigma>0$. If $\rho>1$ and if $\liminf r^{-\rho}\log|f(\pm r)|<\sigma$, then every point of the real axis belongs to the final set of f. An auxiliary result of independent interest is the following theorem on the existence of an interpolating function. (3) Let $f(z)=\sum a_nz^n$ be an entire function, of growth not exceeding order ρ, mean type, and not identically zero. Let $\limsup_{x\to+\infty} x^{-\rho}\log|f(x)|=-\alpha<0$. Then f is of order ρ and positive type σ, with $\rho>\frac{1}{2}$ and $\alpha\leqq\sigma$; there exists a meromorphic function ψ, regular for $\Re(z)\geqq 0$, such that $\psi(n)=(-1)^n a_n\Gamma(1+n/\rho)$, which is real on the real axis

if f is; the poles of ψ are simple and lie at the non-integral values of $-n\rho$ $(n=1, 2, \cdots)$; $T(r, \psi)=O(r)$; and

$$\limsup y^{-1}|\psi(\pm iy)| \leqq \pi(1-\tfrac{1}{2}(1+\omega)/\rho),$$

where $\omega=(2/\pi)\sin^{-1}(\alpha/\sigma)$. Another theorem used in the proof of (1) is a result of Boas and Pólya [Duke Math. J. 9 (1942), 406–424; MR 3,292] on functions satisfying $f^{(\lambda)}(x)f^{(\lambda+2)}(x)<0$. This lead the author to investigate whether there are similar results applying to functions which are not necessarily real. (4) Let $f(z)$ be regular for $|z|<1$ and let $f(z)f''(z)\neq 0$, and $|f(0)f''(0)/\{f'(0)\}^2-1|\geqq\kappa$, where $\kappa<1/e$. Then there are constants A and B such that $|f'(0)/f(0)|\leqq A\kappa^{-1}\log(\kappa^{-1})$ and $|f'(z)/f(z)|<B\kappa^{-3/2}$ for $|z|\leqq 1/A$. (5) If $f(z)$ is regular in a neighborhood of z_0, call a sequence $\{\lambda_k\}$ critical if there exist positive numbers c, κ such that, for every λ_k, $|z-z_0|<c$ implies

$$f^{(\lambda)}(z)f^{(\lambda+1)}(z)f^{(\lambda+2)}(z)\neq 0,$$

$$|f^{(\lambda)}(z_0)f^{(\lambda+2)}(z_0)/\{f^{(\lambda+1)}(z_0)\}^2-1|\geqq\kappa.$$

If $\{\lambda_k\}$ is critical and $\lambda_k-\lambda_{k-1}=O(1)$, f is an entire function of exponential type; if $\lambda_k-\lambda_{k-1}=O(\lambda_k^{\delta})$, $0<\delta<1$, then f is entire, of order at most $1/(1-\delta)$; if $\lambda_k-\lambda_{k-1}=o(\lambda_k)$ then f is entire. (6) Theorem (2) follows from the more general result that if f satisfies a differential equation of the specified kind, is regular at z_0, but z_0 is not a point of the final set of f, then either f is an entire function of exponential type or

$$\lim f^{(n+1)}(z_0)/f^{(n)}(z_0)=\infty,$$

$$\lim f^{(n)}(z_0)f^{(n+2)}(z_0)/\{f^{(n+1)}(z_0)\}^2=1.$$

R. P. Boas, Jr. (Evanston, Ill.).

de La Vallée Poussin, Ch. Théorème de Picard et structure des fonctions: rectifications et compléments. Ann. Soc. Sci. Bruxelles. Sér. I. **70** (1956), 81–86.

The author points out some errors in an earlier paper [same Ann. **69** (1955), 37–49; MR **17**, 601]. These are essentially the same as were indicated independently by the present reviewer in his review of that paper. The author also adds some comments on decompositions of the z-plane associated with functions having Picard exceptional values. *J. A. Jenkins* (Notre Dame, Ind.).

Schubart, Hans. Zur Wertverteilung der Painlevéschen Transzendenten. Arch. Math. **7** (1956), 284–290.

The author investigates the distribution of values of the solutions of the differential equations $w''=6w^2-6z$ and $w''=2w^3+zw+C$. The methods used are based on Nevanlinna's theory of meromorphic functions. *Z. Nehari.*

Rudin, Walter. Boundary values of continuous analytic functions. Proc. Amer. Math. Soc. **7** (1956), 808–811.

The following strengthened form of a result due to P. Fatou [Acta Math. **30** (1906), 335–400] is proved by the author: Let E be a closed subset of the unit circumference C with measure zero and let Φ be a continuous function defined on E; then there exists a function f analytic in the open unit disc and continuous in the closed unit disc, which is an extension of Φ. According to F. and M. Riesz [C. R. 4ième Congrès Math. Scand., Stockholm, 1916, Almqvist-Wiksell, Uppsala, 1920, pp. 27–44] E cannot be assumed here to be a subset of C whose closure has positive measure. Moreover, if the values of Φ on E belong to a two-cell T, then the range of f can also be restricted to T. Finally, an application of the author's theorem is made to show that a condition in his previous paper [Duke Math. J. **20** (1953), 449–457; MR **15**, 21] cannot be omitted. *A. Rosenthal* (Lafayette, Ind.).

Arsove, Maynard G. Some criteria for normality of families of continuous functions. Comm. Pure Appl. Math. **9** (1956), 299–305.

In the usual characterization of normal families of continuous functions the author replaces the condition of equicontinuity by another condition which measures the deviation from harmonicity. Let Ω be a region in the plane, K a compact subset of Ω, w a continuous function on Ω, r a positive number less than the distance from K to the boundary of Ω, and let $\alpha_r w(z)$ denote the areal mean of w over a neighborhood of radius r about z. The author sets:

$$\Lambda_w{}^K(r)=\sup_{z\in K}|w(z)-\alpha_r w(z)|$$

and, if $\mathfrak{F}$ is a family of such functions, w,

$$\Lambda_{\mathfrak{F}}{}^K(r)=\sup_{w\in\mathfrak{F}}\Lambda_w{}^K(r).$$

Then he proves the theorem: The family $\mathfrak{F}$ is normal if and only if for all compact subsets K of 1) $\mathfrak{F}$ is bounded on K and 2) $\lim_{r\to 0}\Lambda_{\mathfrak{F}}{}^K(r)=0$. He shows also that the condition of boundedness in 1) can be replaced by L^p boundedness of $\mathfrak{F}$ on K for $p\geqq 1$. The author then applies his theorems to subharmonic functions and δ-subharmonic functions (i.e. differences of subharmonic functions), summarizing the main results of a previous paper [Proc. Amer. Math. Soc. **7** (1956), 115–126; MR **17**, 1197].

A. Rosenthal (Lafayette, Ind.).

Evgrafov, M. A. The method of near systems in the space of analytic functions and its application to interpolation. Trudy Moskov. Mat. Obšč. **5** (1956), 89–201. (Russian)

It is well-known that if a set of elements is complete, or forms a basis, in a linear topological space, the same is true for any other set whose elements are sufficiently close to the elements of the first set. This paper develops this principle in detail for the space $\mathfrak{A}(D)$ of analytic functions in a given simply-connected domain D with simply-connected complement, with the topology of uniform convergence on compact sets; and also for some spaces of entire functions.

The first chapter contains general properties of the space $\mathfrak{A}(D)$, and of the spaces $\mathfrak{A}(\Phi, D)$ of entire functions, defined as follows: Let $\Phi(z)=\sum_0^\infty z^n/m_n$, $m_n{}^{1/n}\to\infty$; then $\mathfrak{A}(\Phi, D)$ consists of the entire functions F representable in the form

$$F(z)=(2\pi i)^{-1}\int_C \Phi(z\zeta)f(\zeta)d\zeta,$$

with $f\in\mathfrak{A}(D')$ (D' is the complement of D), where $F_n\to F$ means that $f_n\to f$ in $\mathfrak{A}(D')$. The general forms of linear continuous functionals on these spaces are found. Some general theorems about complete systems and bases are given.

The second chapter contains more special criteria. For example, if $\{\phi_n\}$ is complete in $\mathfrak{A}(D)$, $M_n=\max|\phi_n(z)|$ for $z\in D$, and $\phi_n{}^{(1)}(z)=\phi_n(z)+\sum_{k=0}^\infty \varepsilon_{n,k}\phi_k(z)$, then $\{\phi_n{}^{(1)}\}$ is complete if $\sum_{k=0}^\infty|\varepsilon_{n,k}|M_k<\theta M_n$ $(\theta<1)$. There are theorems about operators which transform a complete set or a basis into a complete set or a basis. Then a different measure of nearness is introduced and applied to $\mathfrak{A}(\Phi, D)$, where D is the finite plane. The Gončarov polynomials $P_n(z)$ are defined by $P_n{}^{(k)}(\lambda_k)=0$ $(k<n)$, $P_n{}^{(n)}(z)=1$, where $\{\lambda_n\}$ is a given sequence of points. Their properties are investigated, especially under variation of the λ_n, and conditions are given for the "nearness" of two sets of Gončarov polynomials with different $\{\lambda_n\}$.

In Chapter 3 the results of Chapter 2 are applied to entire functions. Define

$$\Gamma_\rho(z)=\prod(1+c_\rho z/n^{1/\rho}),\ c_\rho\rho=\pi^{-1}\sin\pi\rho.$$

A region G_ρ is found so that if $F\in\mathfrak{A}(\Gamma_\rho, G_\rho)$ then F is represented by its Gončarov series when $\lambda_n=n^{1/\rho}(\rho<1)$, or even when $\lambda_n\rho-n=o(1)$. There are analogous results when $\rho>1$. Numerous other results are obtained. For $\rho<1$, converse results are established and some properties of the sequence $\{F^{(n)}(cn^{1/\rho})\}$ are obtained when F is of order ρ.

Chapter 4 discusses systems which are close to incomplete systems or close to complete systems which are not bases. The discussion is based on Fredholm theorems for operators on $\mathfrak{A}(D)$. As examples of the theorems on completeness we quote the following. Let $P(z)$ be regular in $|z|<R$ and let $\phi_n(z)=z^nP(z)+z^{n+1}\varepsilon_n(z)$, where $\lim_{n\to\infty}\max_{|z|<r}|\varepsilon_n(z)|=0$, $r<R$. Then the number of linearly independent functions Φ, regular for $|z|\geqq r_0$, and satisfying

$$\int_{|z|=r_0}\phi_n(z)\Phi(z)dz=0\ (n=0, 1, 2, \cdots),$$

is equal to the number of zeros of $P(z)$ in $|z|<r_0$. Furthermore, $\{\phi_n\}$ is complete in any simply-connected region lying in $|z|<R$ and containing no zeros of $P(z)$. Applications are given to the problem of the existence of a non-trivial entire function F of prescribed growth with $F^{(n)}(n^{1/\rho})=0$. *R. P. Boas, Jr.* (Evanston, Ill.).

Maschler, Michael. Minimal domains and their Bergman kernel function. Pacific J. Math. **6** (1956), 501–516.

The paper is concerned with the class $\mathfrak{L}$ of domains D over the space of n complex variables C^n that can be mapped pseudo-conformally on bounded open subsets of the C^n. $D\in\mathfrak{L}$ is called a "minimal domain with center $t\in D$" (we write $D\in M_t$) if it has smallest euclidean volume among all domains that map pseudo-conformally onto D with Jacobian$=1$ at t. (It is always assumed that t is not on a branch manifold of D.) The author shows: Any domain in $\mathfrak{L}$ can be mapped onto a minimal domain such that an arbitrary point goes into the center t and the Jacobian is one at t. Necessary and sufficient that $D\in M_t$ is that $K_D(z, t)\equiv$const holds, where K_D is the Bergman kernel function of D. The value of the constant is V^{-1}, where V is the euclidean volume of D. Also if $K(t, \bar t)=V^{-1}$, then $D\in M_t$, and $K_D(z, \bar z)$ takes its minimum in $D\in M_t$ at and only at t. If $D\in M_t$ and f holomorphic in D and square integrable, then $f(t)=V^{-1}\int_D f(z)d\omega$ holds ($d\omega$ the euclidean volume element). If $D\in M_0$, then $\lambda D=\{z|z'=\lambda z, z\in D\}\in M_0$. If $D\in M_t$ and if the points of D can be separated by functions holomorphic in D, then D cannot have more than one center. The product of minimal domains is a minimal domain such that the product of the centers is the center of the product. Besides the "minimal domains" S. Bergman has introduced the "representative domains". The author shows: Necessary and sufficient that D is at the same time a minimal domain and a representative domain with the same center is, that the Bergman kernel function satisfies relations $\partial/\partial\bar t_j K_D(z, \bar t)=\sum_{\nu=1}^n A_{j\nu}(z_\nu-t_\nu)$, $j=1, \cdots, n$, where the A_{ij} are constants and $|A_{ij}|\neq 0$. There exist domains that do not satisfy these relations. Finally, if Δ is a simply connected domain in the complex w-plane, then

$$[K\Delta(w, \bar w)]^{-3}\begin{vmatrix} K_\Delta(w, \bar w) & \dfrac{\partial K_\Delta(w,\bar w)}{\partial\bar w} \\ \dfrac{\partial K_\Delta(w, \bar w)}{\partial w} & \dfrac{\partial^2 K_\Delta(w, \bar w)}{\partial w\partial\bar w}\end{vmatrix}=\text{const}$$

holds, and a corresponding relation is derived for several variables. — Most of the proofs are based on properties of the Bergman kernel function. *H. J. Bremermann.*

Lewy, Hans. On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables. Ann. of Math. (2) **64** (1956), 514–522.

The author considers the linear differential equation $\sum_{i=1}^3 A_i\,\partial u/\partial\alpha_i=0$, where α_1, α_2, α_3 are real variables and the A_i complex valued functions of the α_i. It is assumed that the determinant

$$D=\begin{vmatrix} A_1, \bar A_1, \sum A_i\partial\bar A_1/\partial\alpha_i-\sum\bar A_i\partial A_1/\partial\alpha_i \\ A_2, \bar A_2, \sum A_i\partial\bar A_2/\partial\alpha_i-\sum\bar A_i\partial A_2/\partial\alpha_i \\ A_3, \bar A_3, \sum A_i\partial\bar A_3/\partial\alpha_i-\sum\bar A_i\partial A_3/\partial\alpha_i\end{vmatrix}$$

is different from zero in a neighborhood of a point of the α-space. It is further postulated that two continuously differentiable complex solutions x and y exist such that the rank of the matrix $\partial(x, y)/\partial\alpha_j$ is 2. $x(\alpha)$, $y(\alpha)$ describe a smooth surface S in the space of two complex variables. The condition $D\neq 0$ turns out to be equivalent to the statement that S is strictly (totally) pseudo-convex. Any other solution U can be considered as a function on S. The author shows: Any solution U (with continuous derivatives) can be continued into one side of S as a holomorphic function of x and y, the side of continuation depending upon S but not upon U. Conversely every holomorphic function of x and y existing on one side of S and having continuous derivatives upon approach to S satisfies, as function of α_1, α_2, α_3 the above differential equation. *H. J. Bremermann* (Princeton, N.J.).

Wintner, Aurel. On the process of successive approximation in initial value problems. Ann. Mat. Pura Appl. (4) **41** (1956), 343–357.

The author determines lower bounds for the greatest possible value α_f^* such that the initial value problem $dy/dx=f(x, y)$, $y(0)=0$ with $f(x, y)$ real-valued and continuous in the rectangle

$$0\leqq x\leqq a,\ -b\leqq y\leqq b \tag{1}$$

has at least one solution $y=y(x)$ satisfying $|y(x)|\leqq b$ on $0\leqq x\leqq\alpha_f^*$.

For this purpose he considers primarily lower bounds for α_f^*, that is the greatest possible value such that the process of successive approximations

$$y_{n+1}(x)=\int_0^x f(t, y_n(t))dt,$$

where $y_0(x)\equiv 0$, gives $|y_n(x)|\leqq b$ for $0\leqq x\leqq\alpha_f^*$ and any n.

These lower bounds are obtained by replacing $f(x, y)$ by $F(x, y)=\max|f(x, s)|$ for $-|y|\leqq s\leqq|y|$ or other (greater) functions $\varphi(x, y)$ which, in the rectangle (1) are non-negative, continuous, even in y, and do not decrease, when x is fixed and $|y|$ increases.

In this manner the author improves and generalizes, for instance, a well-known result of Lindelöf.

Analogous considerations are made in the complex field, where his criterion of "subordination", rather than of "majorisation", leads the author finally to the following statement:

If both $f(z, w)$ and $g(z, w)$ are regular on a dicylinder ($|z|<a$, $|w|<b$) and if, at every point (z, w) of the latter

$$|f(z, w)|\leqq g(|z|, |w|),$$

then the solution $w(z)$ of the initial value problem

$$\frac{dw}{dz}=f(z, w),\ w(0)=0,$$

is regular on every circle $|z|<\text{const}$, on which the solution $w(z)$ of the initial value problem

$$\frac{dw}{dz}=g(z, w),\ w(0)=0$$

happens to be regular. *M. J. De Schwarz* (Rome).

Stark, J. M. On distortion in pseudo-conformal mapping. Pacific J. Math. **6** (1956), 565–582.

Let C^2 be the space of two complex variables, B a domain in C^2 and $\mathfrak{L}^2(B)$ the class of in B square integrable holomorphic functions. The author is concerned with minima of the integral $\int_B |f|^2 d\omega$ ($d\omega$ the euclidean volume element of C^2) for f in various subclasses of $\mathfrak{L}^2(B)$ defined by "auxiliary conditions" of f. For instance let λ_B^1 be the minimum of the integral under the condition $f \in \mathfrak{L}^2(B)$, $f(t)=1$ for a fixed point $t \in B$, and λ_{BB}^1 the minimum for $f \in \mathfrak{L}^2(B)$, $f(t)=1$ and $\int_B f d\omega=0$. Then $(\lambda_B^1)^{-1}=(\lambda_{BB}^1)^{-1}+(\text{vol } B)^{-1}$ holds, where vol B is the euclidean volume of B. Five more relations of this type are proved involving different minima ("λ's") belonging to various other "auxiliary conditions".

By introducing "domains of comparison" I and A, such that $I \subset B \subset A$, inequalities are obtained for each of the previous equalities estimating the λ's with respect to B by those with respect to I and A. These relations are then applied to obtain "distortion theorems" between the euclidean line element dS^2 and the ds^2 of the pseudo-conformally invariant Bergman metric generated by the Bergman kernel function (which can be expressed by λ's). For instance it is shown: $q_1 \leq ds^2/dS^2 \leq q_2$, where q_1 and q_2 are somewhat complicated expressions involving several different λ's. Other estimates for ds^2 alone and other pseudo-conformal invariants related to the Bergman kernel function are also obtained and finally the results are applied by using spheres, eccentric spheres, polycylinders, etc. as domains of comparison for which some of the λ's can be calculated explicitly. The paper utilizes methods developed by S. Bergman. *H. J. Bremermann.*

Agrusti, Giovanni. Su alcune proprietà delle derivate parziali prime destre e sinistre dei raggi associati di convergenza di una serie di potenze ad n variabili complesse. Giorn. Mat. Battaglini (5) **4(84)** (1956), 69–73.

The numbers $y_1, y_2, \cdots, y_n$ are called the associated radii of convergence of the power series

$$\sum_0^\infty r_1 r_2 \cdots r_n a_{r_1 r_2 \cdots r_n} x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$$

in n complex variables if the power series is absolutely convergent for $|x_1|<y_1, |x_2|<y_2, \cdots, |x_n|<y_n$ and absolutely divergent for $|x_1|>y_1, |x_2|>y_2, \cdots, |x_n|>y_n$. Faber [Math. Ann. **61** (1905), 289–324] (in the case $n=2$) showed that the set of points (y_1, y_2) constitutes a curve $f(y_1, y_2)=0$ having various properties. For example, $f(y_1, y_2)=0$ is the solution curve of a second order differential equation. In an earlier paper [Giorn. Mat. Battaglini (5) **3(83)** (1955), 223–296; MR **17**, 1071] the present author extended much of the work of Faber to n complex variables. In the present paper he is concerned with the equations $f(y_1, y_2, \cdots, y_n)=0$ and in particular with inequalities which hold between the quantities $\partial y_i/\partial y_n$. The nature and complicated character of these results precludes their presentation here. *V. F. Cowling.*

See also: Levinson, p. 468; Komatu, p. 475; Heins, p. 478; Jurkat, p. 479; Griffith, p. 481; Suvorov, p. 496; Phillips, p. 525; Woods, p. 529; Legendre, p. 530.

Geometrical Analysis

★ **Borg, Sidney F. An introduction to matrix tensor methods in theoretical and applied mechanics.** J. W. Edwards, Inc., Ann Arbor, Mich., 1956. iv+202 pp. $4.75.

This book is intended principally as a textbook for engineers at the senior and graduate level. After brief reviews of topics in matrix algebra, vector analysis, and complex variable theory, the concept of a tensor is introduced and is exploited first in the derivation of the basic equations of the theory of elasticity. Applications to some classical problems in this field are then given. Concise treatments of the elementary parts of the theory of viscous flow and of the theory of plasticity follow, and the book concludes with chapters dealing with curvilinear coordinates and dimensional analysis.

Mathematical subtleties are purposely ignored, and occasionally significant limitations on the validity of formal results are not displayed. Also, some readers may encounter a few unfamiliar terms which are undefined. {However, the expressed dual aims of the author to supply the engineering student with tensor and matrix methods as a working tool and to present a readable unified survey of the principles of the several fields of applied mechanics in a book of modest size — would appear to have been satisfactorily attained.} *F. B. Hildebrand.*

Kuranishi, Masatake. On E. Cartan's prolongation theorem of exterior differential systems. Amer. J. Math. **79** (1957), 1–47.

In this very brilliant paper the author gives a necessary and sufficient condition for an exterior differential system to become a general solution (involutive solution) after a finite number of prolongations. The concept of prolongation was introduced by E. Cartan precisely for this purpose, but Cartan never succeeded in proving any general theorems concerning when a solution becomes regular on prolongation. However, Cartan did calculate numerous concrete examples of exterior systems and showed that these systems had general solutions after a finite number of prolongations. Cartan's proof of the local imbedding of riemann manifolds in euclidean spaces of sufficiently high dimension was just such a calculation. The author's general theorem should now make the exterior differential systems an extremely valuable tool for dealing with existence theorems for differential or partial differential equations on manifolds in which real analytic data is given.

The paper itself is divided into three parts. Part I is devoted to a summary of the general theory of exterior differential systems, in general without proofs; Part II deals with the definition and the study of normal exterior systems; in Part III the author proves the main theorem:

Let us be given a normal system with p independent variables and let us consider the lth prolongation space. Let ρ^l denote the natural projection of the lth prolongation space into the original space. Let x be a regular point in the original space. Then there exists an integer $l_1(x)$ satisfying the following conditions: if y, $\rho^l(y)=x$, is a p-integral point in the lth prolongation space and if

$l \geq l_1(x)$ then our prolonged system is general at y if and only if y is a normal point in the lth prolonation space.

The proof of this fundamental theorem is too lengthy and technical to give even an adequate outline in a review. The basic idea, however, is to reduce the problem to one involving polynomial ideals and here the author is able to apply classical theorems. *L. Auslander.*

See also: Bulešev, p. 485.

Functions with Particular Properties

Betts, D. D.; Bhatia, A. B.; and Wyman, Max. Houston's method and its application to the calculation of characteristic temperatures of cubic crystals. Phys. Rev. (2) **104** (1956), 37–42.

A method is given of determining type-α Kubic harmonics K_m (homogenous polynomials of degree $2m$ satisfying Laplace's equation, invariant under the operations of cubic symmetry, and orthogonal on the unit sphere; other types are invariant only for certain of the symmetry operations), and those with $m \leq 6$ are evaluated explicitly. Functions with cubic symmetry can then be approximated by

$$I(\theta, \phi) = \sum a_m K_m(\theta, \phi),$$

where the coefficients a_m can be evaluated from a knowledge of $I(\theta, \phi)$ along certain directions only. Houston's approximation [Rev. Mod. Phys. **20** (1948), 161–165] consists in using $K_0 \equiv 1$, $K_1 \equiv 0$, K_2 and K_3, and the directions (100), (110) and (111). Seven closer approximations, obtained by using one of, two of, or all the functions K_4, K_5, K_6 and the directions (210), (211), (221) in addition are considered, and applied in particular to evaluating integrals of the type

$$J = \iint I(\theta, \phi) \sin\theta \, d\theta d\phi$$

arising in the calculation of characteristic temperatures of cubic crystals. The results are compared with those obtained by numerical integration by Blackman [Proc. Roy. Soc. London. Ser. A. **164** (1938), 62–79], and it is found that for not markedly anisotropic crystals $[1/4 < 2c_{44}/(c_{11}-c_{12}) < 4$, where the c's are elastic constants] the error is less than 1% for the approximation with all six terms included. For the markedly anisotropic alkali metals, for which the ratio above approaches 8, the error may be as much as 10%. *A. J. C. Wilson* (Cardiff).

Komatu, Yûsaku. Further supplement to "On transference of boundary value problems". Kōdai Math. Sem. Rep. **8** (1956), 1–8.

Let D be a domain in the complex z-plane bounded by a rectilinear or circular slit. If $f(z)$ and $g(z)$ are analytic functions in D such that the boundary values of $\mathrm{Re}\{f(z)\}$ coincide with the values of the normal derivative of $\mathrm{Re}\{g(z)\}$ at the boundary of D, the author derives the explicit formulas connecting $f(x)$ and $g(z)$. While in two previous papers [same Rep. **1954**, 71–80, 97–100; MR **16**, 698] these relations were obtained with the help of Poisson's formula, the author is now able to give a direct proof. *Z. Nehari* (Pittsburgh, Pa.).

Grauert, Hans; und Remmert, Reinhold. Plurisubharmonische Funktionen in komplexen Räumen. Math. Z. **65** (1956), 175–194.

The results of this paper (on continuation of plurisubharmonic functions defined in general complex spaces) have been announced in C. R. Acad. Sci. Paris **241** (1955), 1371–1373 [MR **17**, 662]. *H. J. Bremermann.*

Langenbah, A. Approximate solution of the biharmonic equation for a trapezoidal region. Vestnik Leningrad. Univ. **11** (1956), no. 13, 119–123. (Russian)

In this paper a method of M. G. Slobodiansky [Prikl. Mat. Meh. (N.S.) **3** (1939), no. 1, 75–82] is applied to the biharmonic equation for a trapezoidal region. The region is subdivided by equidistant straight lines (distance h) parallel to the two parallel sides of the trapezoid and the biharmonic equation is replaced by a system of fourth order ordinary differential equations which is exact up to terms of order h^4 and can be integrated in closed form. Using the boundary conditions to determine the free constants one can find thus an approximate solution of the problem on the straight lines.

The author then describes an interpolation method similar to a method of V. N. Faddeeva [Trudy Mat. Inst. Steklov. **28** (1949), 73–103; MR **12**, 362] which allows to extend the approximate solution of the biharmonic equation obtained by the previous method to the entire trapezoid. For this the same subdivision with straight lines is used as before. The solution u of the problem is assumed to have the form

$$u_\nu(x, y) = \sum_{k=0}^{6} a_{\nu,k}(x)(y - y_\nu)^k$$

near the νth straight line $y = y_\nu$. The coefficients $a_{\nu,k}(x)$ are determined using the partial differential equation, the previously obtained approximate solutions and the boundary conditions.

Finally, the author mentions that the method is also applicable to the more general equation:

$$\frac{\partial^4 u}{\partial x^4} + a\frac{\partial^4 u}{\partial x^2 \partial y^2} + b\frac{\partial^4 u}{\partial y^4} = f(x, y)$$

and in some case also to regions with rounded contours. *U. W. Hochstrasser* (Washington, D.C.).

Brelot, M. Nouvelle démonstration du théorème fondamental sur la convergence des potentiels. Ann. Inst. Fourier, Grenoble **6** (1955–1956), 361–368.

In this note the author gives a new proof of the following fundamental result due to H. Cartan [Bull. Soc. Math. France **73** (1945), 74–106; MR **7**, 447]. Let $\mathcal{D}$ be an open set in E^n $(n \geq 2)$, and let $\mathcal{F}$ denote a non-empty family of non-negative functions U defined and superharmonic in $\mathcal{D}$. Then $\underline{U} = \inf [U | U \in \mathcal{F}]$ is quasi-superharmonic in $\mathcal{D}$, i.e., $\underline{U}$ differs from a superharmonic function on a set at most of exterior capacity zero. Where Cartan's proof depended upon his study of the energy integral the present author's proof depends upon the compactness, in the "vague" topology, of a certain set of non-negative measures, and upon Choquet's recent theory of capacitability [Ann. Inst. Fourier, Grenoble **5** (1953–1954), 131–295; MR **18**, 295]. *M. Reade.*

Shapiro, Victor L. Generalized Laplacians. Amer. J. Math. **78** (1956), 497–508.

The author considers functions f defined in a bounded domain R in euclidean n-space, and defines $\Delta_r f$, the rth generalized Laplacian of f. To avoid introduction of a great deal of notation, let us merely remark that the definition of $\Delta_r f$ is based on a finite Taylor expansion of f,

and that $\Delta_r f=\Delta^r f$ if f has $2r$ continuous derivatives, where Δ^r is the usual Laplace operator, iterated r times. For $r=1$, upper and lower generalized Laplace operators are also considered. Theorem 1 asserts that if f and $\Delta_j f$ $(1\leqq j\leqq r-1)$ are continuous in R, if Z is a compact set of capacity 0, and if $\Delta_r f(x)$ exists in $R-Z$ and is integrable on R, then in every subdomain R' whose closure lies in R,

$$f=\underbrace{P*\cdots*P}_{r}*\Delta_r f+H_r,$$

where $\Delta^r H_r(x)=0$ in R', $P(x)=-(2\pi)^{-1}\log|x|^{-1}$ if $n=2$, $P(x)=-c_n|x|^{2-n}$ if $n\geqq 3$ $(c_n>0)$, and the convolution operation $*$ is defined in R' by $(f*g)(x)=\int_{R'} f(x-y)g(y)dy$. The case $r=1$ is due to the reviewer (actually under slightly weaker hypotheses) [Trans. Amer. Math. Soc. **68** (1950), 278–286; MR **11**, 663]. The proof of Theorem 1 uses the case $r=1$ and proceeds by induction. Theorem 2 deals with the case $r=1$ and extends the above mentioned result of the reviewer to the n-dimensional torus. It is then applied to yield a uniqueness theorem concerning Abel summable multiple trigonometric series.

W. Rudin (Rochester, N.Y.).

Sendov, Blagovest. On a certain class of regular-monotone functions. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 27–30. (Russian)

Let $\{\varepsilon_n\}_{n=0}^{\infty}$ be a sequence consisting entirely of ± 1's. Let there be a positive integer q such that $\varepsilon_{n+q}=\varepsilon_n$ $(n=0, 1, 2, \cdots)$, and suppose that q is fixed as the least such integer. Let $x_n=1$ if $\varepsilon_n\neq\varepsilon_{n+1}$ and $x_n=0$ if $\varepsilon_n=\varepsilon_{n+1}$. Let $\tau_n=-1$ if $\varepsilon_n\neq\varepsilon_{n+1}$ and $\tau_n=1$ if $\varepsilon_n=\varepsilon_{n+1}$. Let K_ε be the class of all real-valued functions f defined on $]0, 1[$ such that $f^{(n)}$ exists and $\varepsilon_n f^{(n)}(x)\geqq 0$, $0<x<1$ and $n=0, 1, 2, \cdots$. Let

$$P(x)=\int_{x_0}^{x}\tau_0\,dt_1\int_{x_1}^{t_1}\tau_1 dt_2\cdots\int_{x^{\nu-1}}^{t_{\nu-1}}\tau_{\nu-1}dt\quad(\nu=0, 1, 2, \cdots).$$

Theorem. Every function f in K_ε can be represented in the form

$$f(x)=\sum_{\nu=0}^{\infty} a_\nu P_\nu(x)+Ag(x),$$

where $a\geqq 0$, $A\geqq 0$, and g is a function in K_ε with the following properties: $g^{(q)}=\alpha^q g$ $(\alpha\geqq 1)$;

$$\lim_{x\to x_n} g^{(n)}(x)=0\quad(n=0, 1, 2, \cdots);$$

g is irreducible in the sense of Tagamlicki [Ann. Univ. Sofia Fac. Sci. Phys. Math. Livre 1. **48** (1953/54), 69–85; MR **17**, 767]. The Theorem is due to S. Bernšteĭn for absolutely monotone functions [Collected Works, vol. 2, Izdat. Akad. Nauk SSSR, 1954, pp. 100, 106; see also vol. 1, 1952, p. 35; MR **16**, 433; **14**, 2]. No proofs are given.

E. Hewitt (Seattle, Wash.).

See also: Edrei, p. 471; Arsove, p. 472; Greco, p. 486; Aczél, p. 488; Wright, p. 493; Wegner, p. 525.

Special Functions

Munschy, Gilbert. Sur quelques relations de récurrence entre certains polynomes d'Appell et Kampé de Fériet. C. R. Acad. Sci. Paris **243** (1956), 767–770.

The polynomials in question are

$$F_{mn}(x, y)=\frac{1}{m!n!}\frac{\partial^{m+n}}{\partial x^m\partial y^n}[x^m y^n(1-x-y)^{m+n}].$$

The author defines certain differential operators, $\mathscr{A}_i$ and $\mathscr{B}_i$ and expresses $\mathscr{A}_i F_{mn}$ and $\mathscr{B}_i F_{mn}$ as finite linear combinations of the F_{rs}. The results are said to be useful in connection with the Schrödinger equation of two electrons in a central field.

A. Erdélyi (Jerusalem).

Heegner, Kurt. Reduzierbare Abelsche Integrale und transformierbare automorphe Funktionen. Math. Ann. **131** (1956), 87–140.

In dieser Arbeit wird die Beziehung zwischen die Theorie der Reduktion Abelscher Integrale und die Lösung diophantischer Fragestellungen betrachtet. Die transcendente Theorie der einfachen Reduzierbarkeit der Abelschen Integrale wird dabei auf die Aufgabe der Darstellung einer positiven natürlichen ganzen Zahl durch eine ternäre Hermitische Form

$$H=u\bar{v}+\bar{u}v+w\bar{w}=H(u, v, w)$$

zurückgeführt. Die Auffindung der unimodularen Transformationen des Periodenschemas wird auf die der Automorphismen der Hermiteschen Form H zurückgeführt. Es ergibt sich dabei dass die Automorphismengruppe von H vollständig erzeugt wird durch die drei Substitutionen V_1, $V_2(1)$ und $V_3(i+1)$ wo:

$$V_1: u=v',\ v=u',\quad V_2(m): u=u'+imv\ (m \text{ ganz rational}),$$

$$V_3(c)\begin{cases} u=u'-\frac{1}{2}c\bar{c}v'-\bar{c}w',\\ v=v',\\ w=cv'+w'.\end{cases}$$

Insbesondere wird gezeigt, dass die Anzahl der reduzierten Darstellungen von $-m$ durch H endlich ist. Die Darstellungstheorie ganzer Zahlen durch H findet dadurch ihren Abschluss, dass auch die Darstellung binärer Hermitescher Formen durch H in Betracht gezogen wird. Es ergibt sich u.a.:

Die indefinite Hermitesche Form $(1, 0, -n)$ hat nur eine Darstellung durch H. Sämtliche Darstellungen einer positiven ganzen Zahl $n\equiv 0, 2, 3 \pmod 4$ durch H sind einander äquivalent.

Bei den binären Hermiteschen Formen von der Determinante $D=-2$ und dem Teiler $(1+i)$ ist die reproduzierende Gruppe oktaedrisch. Die orthogonale Hermitesche Form zeigt sich äquivalent mit

$$H_2=(1+i)x\bar{y}+(1-i)\bar{x}y+z\bar{z}.$$

Die Gruppe von H_2 wird in den Variabeln von H durch $V_2(2)$, (6), (8) vollständig erzeugt und die Transformationsgruppe modulo $(1+i)$ als Untergruppe der unimodularen Substitutionen durch V_0, V_1, $V_2(2)$, (8).

S. C. van Veen (Delft).

Roelcke, Walter. Über die Wellengleichung bei Grenzkreisgruppen erster Art. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1953/1955**, 159–267 (1956).

Automorphic wave functions, introduced by H. Maas [Math. Ann. **121** (1949), 141–183; MR **11**, 163] are (non-analytic) eigenfunctions $\phi(x, y)$ belonging to the operator $\Delta=y^2(\partial^2/\partial x^2+\partial^2/\partial y^2)$ and invariant under a Fuchsian group Γ in the upper half plane, which has a fundamental domain with only cusps on the real axis. Thus the Eisenstein-type series $E=\sum y^{is}|m_1\tau+m_2|^{-s}$ $(\tau=x+iy$, summed over an equivalence class of cusps $-m_2/m_1$ under the group $\Gamma)$, satisfies $\Delta E=-[\frac{1}{4}s(2-s)]E$. When Γ is the principal congruence subgroup modulo Q the spectrum is continuous and a complete set of eigenfunctions can be constructed from these E-functions. The multiplicity would equal the number of different equivalent classes

of cusps. For Hecke's group, the situation is less conclusive but estimates are given, based on Hilbert-space techniques, (Green's functions, Hellinger's integrals, etc.) The concluding remarks have bearing on the correspondence with Dirichlet series.

H. Cohn.

Maximon, L. C. Two expansions of the Bessel function $K_n(z)$ in terms of $I_n(z)$. Quart. J. Math. Oxford Ser. (2) **7** (1956), 79–80.

The author obtains two expansions of $K_n(z)$ in infinite series involving $I_m(z)$ $(m=0, 1, \cdots)$ by utilising the corresponding expansions (by Neumann and Schläfli respectively) of $Y_n(z)$ in terms of $J_m(z)$.

A. Erdélyi.

Cooke, J. C. Some relations between Bessel and Legendre functions. Monatsh. Math. **60** (1956), 322–328.

Certain infinite integrals involving Bessel functions are expressed in terms of associated Legendre functions in oblate and prolate spheriodal coordinates. Expansions of Bessel functions in series involving Legendre functions are also given. (Author's summary.)

N. D. Kazarinoff.

★**Thorne, R. C. The asymptotic expansion of Legendre functions of large degree and order.** Department of Mathematics, California Institute of Technology, Technical Report 13, Office of Naval Research NR 043–121, 1956. pp. 22–70 (bound with Report 12).

Asymptotic expansions for the Legendre functions $P_n^{-m}(z)$ and $Q_n^{-m}(z)$ are derived under the following conditions: z is an unrestricted complex variable, $0<m<n$, $n\to\infty$, and $\alpha=m/(n+\frac{1}{2})$ is fixed. The behavior of Legendre functions under these assumptions has previously never been investigated. Hobson's definition of the Legendre functions is used.

Three kinds of expansions are found. The first are expansions in terms of exponential functions and are uniformly valid as $n\to\infty$ for $\Re(z)\geqq 0$ except for the strips $|\Im(z)|<\delta$, $\Re(z)<\beta+\delta$, where $\delta>0$ and $\beta=(1-\alpha^2)^{\frac{1}{2}}$. The second are expansions in terms of Airy functions and are uniformly valid as $n\to\infty$ for all z in the z-plane, cut from $+1$ to $-\infty$, except for the strip $|\Re(z)|<\beta+\delta$, $-\delta<\Im(z)\leqq 0$ and except for z in a certain pear-shaped domain surrounding $z=-1$. The third are expansions in terms of Bessel functions of order m whose argument may be large or small. These are uniformly valid as $n\to\infty$ for all z in the cut z-plane except for values of z in the pear-shaped domain surrounding $z=-1$.

N. D. Kazarinoff.

Mitrinović, Dragoslav S. Quelques formules concernant les polynômes de Legendre. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. no. 1 (1956), 20 pp. (Serbo-Croatian. French summary)

L'auteur utilise le fait, que la dérivée d'ordre r d'un polynôme de Legendre P_n peut s'exprimer comme combinaison linéaire de ces polynômes pour calculer

$$\int_{-1}^{+1}(D_x{}^rP_m(x))(D_x{}^sP_n(x))dx,$$

où $D_x{}^r=d^r/dx^r$, et r, m, s, n sont des entiers.

M. Tomić (Beograd).

Gatteschi, Luigi. Una nuova rappresentazione asintotica dei polinomi di Legendre mediante funzioni di Bessel. Boll. Un. Mat. Ital. (3) **11** (1956), 203–209.

The author proves

$$\left(\frac{\sin\theta}{\theta}\right)^{1/2}P_m(\cos\theta)=J_0(u\theta)-\frac{\theta}{24u}J_1(u\theta)+\varepsilon(\theta),$$

where $u=n+\frac{1}{2}$, and $\varepsilon(\theta)=\theta^4O(1)$ if $0<\theta\leqq\pi/(2n)$, $\varepsilon(\theta)=\theta^{5/2}O(n^{-3/2})$ if $\pi/(2n)\leqq\theta\leqq\pi/2$. This result is used to obtain the corresponding asymptotic form for the zeros of Legendre polynomials.

A. Erdélyi (Jerusalem).

Massaro, Giliana. Il termine complementare di una nuova formula per la valutazione asintotica dei polinomi di Legendre. Boll. Un. Mat. Ital. (3) **11** (1956), 433–439.

The author shows that the function $\varepsilon(\theta)$ defined in the preceding review satisfies the inequalities $|\varepsilon(\theta)|<.03\theta^4$, if $0<\theta\leqq\pi/2n$, and $|\varepsilon(\theta)|<.25\theta^{5/2}n^{-3/2}$, if $\pi/2n<\theta\leqq\pi/2$.

N. D. Kazarinoff (Ann Arbor, Mich.).

Ragab, F. M. Some formulae for the product of hypergeometric functions. Proc. Cambridge Philos. Soc. **53** (1957), 106–110.

The integrals of the class

$$\int_0^\infty\cdots\int_0^\infty {}_2F_1\left(\alpha,\beta;\gamma;\frac{-b}{x_1\cdots x_{p-1}}\right)\times\prod_{r=1}^{p-1}[x^{(r-p)/p}{}_2F_1(\alpha,\beta;\gamma;-x_r)]dx_1\cdots dx_{p-1},$$

where $\Re(\alpha)>(p-1)/p$, $\Re(\beta)>(p-1)/p$, $b>0$, and $p=2, 3, \cdots$, are evaluated.

N. D. Kazarinoff.

Chak, A. M. A generalization of Whittaker's integral. Ann. Univ. Lyon. Sect. A. (3) **18** (1955), 27–33.

The author investigates the function defined by the integral

$$z^k\int_{-1}^1 e^{tzt}C_n{}^k(t)(1-t^2)^{k-\frac{1}{2}}dt$$

obtaining a series expansion, recurrence and differentiation formulas, a differential equation, and several operational forms. $C_n{}^k$ is the Gegenbauer function.

A. Erdélyi (Jerusalem).

Carlitz, Leonard. On Jacobi polynomials. Boll. Un. Mat. Ital. (3) **11** (1956), 371–381.

If $f_n{}^{(\alpha,\beta)}$ is a polynomial in x and β of degree n in x satisfying two recursion formulas, one with respect to n, the other with respect to α, and such that $f_n{}^{(\alpha,\beta)}(1)=P_n{}^{(\alpha,\beta)}(1)$, then $f_n{}^{(\alpha,\beta)}=P_n{}^{(\alpha,\beta)}$. Beyond this, many recursion, summation and generating function relations are derived. As the author states, several of the latter were first given by Feldheim [Acta Math. **75** (1943), 117–138; MR **7**, 65].

N. D. Kazarinoff (Ann Arbor, Mich.).

See also: Hodges, p. 459; Tricomi, p. 479; Bhonsle, p. 480; Griffith, p. 481; Bochner, p. 484; Vaněk, p. 488; Slater, p. 515; Wintner, p. 518; Romani, p. 521; Stefaniak, p. 531; Grümm, p. 535; Grümm und Kremser, p. 535; Millsaps and Pohlhausen, p. 538; Gutzwiller, p. 541; Ingarden, p. 542.

Sequences, Series, Summability

Heins, Albert E. The scope and limitations of the method of Wiener and Hopf. Comm. Pure Appl. Math. **9** (1956), 447–466.

A number of problems in hydrodynamics, acoustics and electrodynamics which lead to integral equation of the Wiener-Hopf type

$$\lambda f(x)+\int_0^\infty f(y)K(x-y)dy=g(x)$$

are discussed. Solutions involving complex Fourier transforms and analytic continuation are given.

E. C. Titchmarsh (Oxford).

Klamkin, M. S. On a generalization of the geometric series. Amer. Math. Monthly **64** (1957), 91–93.

Wynn, P. On a procrustean technique for the numerical transformation of slowly convergent sequences and series. Proc. Cambridge Philos. Soc. **52** (1956), 663–671.

One class of transformations to speed the convergence of series is based on fitting the early terms of the series to functions of a simple form and obtaining the sum by proceeding to the limit in the fitted function. For instance the δ^2-process fits the partial sums to the sum of a (finite set of) geometric series. It is here suggested that the partial sums be regarded as rational functions of n, in virtue of Thiele's interpolation formula [cf. L. M. Milne-Thomson, The calculus of finite differences, Macmillan, London, 1933, p. 107]. The rational functions are the convergents to a certain continued fraction and the asymptotic behavior depends only on the ratio of the leading coefficients.

This idea is examined theoretically and exploited practically on various examples, such as the series $\sum(-)^n/n$, $\sum n^{-2}$, and the sequence

$$(1+1/2+\cdots+1/n-\log_e n).$$

The behavior in the case where the partial sums are of the form $s_n=a+bc^n$, and therefore not of the prescribed form, is discussed. *J. Todd* (Washington, D.C.).

Jakimovski, Amnon. A note on Hausdorff transforms. Proc. Amer. Math. Soc. **7** (1956), 803–807.

If $\{t_p\}_0^\infty$ is a linear transform of the sequence $\{s_q\}_0^\infty$ of the form $t_p=\sum_{q=0}^p a_{p,q}s_q$, then $t_p^*=\sum_{q=0}^p a_{p,p-q}s_q$ is called the reverse transform. Let $\mu_0, \mu_1, \cdots$ be a given sequence of constants and let $\Delta\mu_n=\mu_n-\mu_{n+1}$. If

$$a_{p,q}=\binom{p}{q}\Delta^{p-q}\mu_q,$$

then $\{t_p\}$ is said to be the Hausdorff transform of $\{s_q\}$ generated by the sequence $\mu_0, \mu_1, \cdots$; briefly $\{t_p\}$ is the (H, μ_p) transform of $\{s_q\}$. The author shows that if $\{t_p\}$ is the (H, μ_p) transform of $\{s_q\}$ then $\{t_p^*\}$ is the (H, λ_p) transform of $\{s_q\}$ where $\lambda_p=\Delta^p\mu_0$. *I. I. Hirschman.*

Erdös, P.; et Karamata, J. Sur la majorabilité C des suites de nombres réels. Acad. Serbe Sci. Publ. Inst. Math. **10** (1956), 37–52.

With C representing the arithmetic mean transformation C_1, a real sequence $a_1, a_2, \cdots$ is said to be majorable C with the majorability constant $A<\infty$ if there exists a sequence $A_1, A_2, \cdots$ such that $a_k\leqq A_k$ for each k and

$$\lim_{n\to\infty} n^{-1}\sum_{k=1}^n A_k=A.$$

The following and two similar theorems are given. In order that $a_1, a_2, \cdots$ be majorable C, it is necessary and sufficient that (i) as $n\to\infty$, $\sum_{j=n+1}^{n+k} a_j<o(n)$ when $k=o(n)$ and (ii) $W(\varepsilon)=O(\varepsilon)$ where

$$W(\varepsilon)=\limsup_{n\to\infty}\max_{n<n'\leqq(1+\varepsilon)n}\left\{\frac{1}{n}\sum_{j=n+1}^{n'}a_j\right\};$$

moreover if (i) and (ii) hold, then, as $\varepsilon\to 0+$, $\varepsilon^{-1}W(\varepsilon)=o(1)+A^*$, where A^* is the least majorability constant. It is shown how that concept of majorable C and these theorems are related to quadrature theorems, to Tauberian theorems, and to the prime number theorem.

R. P. Agnew (Ithaca, N.Y.).

Tanzi Cattabianchi, Luigi. Criteri di inversione per la convergenza delle successioni dalla convergenza delle successioni perturbate. Riv. Mat. Univ. Parma **6** (1955), 375–388.

New Mercerian theorems are proved. Assuming that all numbers involved are real, let $v_n=x_n+\beta_n X_n$ where

$$X_n=(n+1)^{-1}\sum_{k=0}^n x_k.$$

In many different ways, the author formulates conditions which, together with the hypothesis that $v_n\to 0$, imply the conclusion that $x_n\to 0$. Most of these conditions involve one or both of the sequences B_n and V_n defined by

$$B_n=(n+1)^{-1}\sum_{k=0}^n\beta_k \text{ and } V_n=(n+1)^{-1}\sum_{k=0}^n v_k.$$

Some of the conditions involve separately the integers r for which $x_rX_r\geqq 0$ and the integers s for which $x_sX_s<0$. One of the theorems is the following. If $v_n\to 0$, $\beta_n>0$, $nv_n=o(\beta_n)$ and $\beta_n=o(nB_n(1+n|V_n|)^{-1})$, then $x_n\to 0$.

R. P. Agnew (Ithaca, N.Y.).

Buck, R. Creighton. An addendum to "A note on subsequences". Proc. Amer. Math. Soc. **7** (1956), 1074–1075.

The author [Bull. Amer. Math. Soc. **49** (1943), 898–899; MR **5**, 117] gave the following theorem. A real sequence $\{x_n\}$ must be convergent if there exists a regular matrix T which transforms each subsequence of $\{x_n\}$ into a convergent sequence. The proof began with the observation that "we may restrict ourselves to bounded sequences." In response to queries, the author now treats unbounded sequences and completes the proof of his theorem.

R. P. Agnew (Ithaca, N.Y.).

Nishimura, Isaki. On the approximation by Cesàro mean of Fourier series. Bull. Kyushu Inst. Tech. (Math., Nat. Sci.) no. **1** (1955), 11–18.

Let $\sigma_n^r(x)$ and $\bar\sigma_n^r(x)$ denote the C_r transforms of the Fourier series of $f(x)$ and the conjugate function $\bar f(x)$, where $f(x)$ is continuous over $-\infty<x<\infty$ and has period 2π. If $0<\alpha<1$, $r>-1$, and $\sigma_n^r(x)-f(x)=O(n^{-\alpha})$ uniformly, then $\bar\sigma_n^r(x)-\bar f(x)=O(n^{-\alpha})$ uniformly. If $r>0$ and $\sigma_n^r-f(x)=O(n^{-1})$ uniformly, then $\bar\sigma_n^r-\bar f(x)=O(n^{-1}\log n)$ uniformly and the logarithm cannot be omitted. There are other theorems involving $|\sigma_n^r(x)-f(x)|$, $|\bar\sigma_n^r(x)-\bar f(x)|$ and their integrals $0\leqq x\leqq 2\pi$. The theorems are modifications of theorems which are known for the case in which $r=0$ and the Cesàro transforms reduce to sequences of partial sums. *R. P. Agnew.*

Ghosh, P. K. On (ϕ)-convergent integrals and their application to mathematical physics. Bull. Calcutta Math. Soc. **48** (1956), 33–44.

The author establishes some properties of infinite integrals that are summable by a rather general method. He applies his results to justify some calculations that occur in elementary quantum mechanics and are customarily justified by ad hoc methods. *R. P. Boas, Jr.*

Standish, Charles. A class of measure preserving transformations. Pacific J. Math. **6** (1956), 553–564.

To each permutation π of the set of all positive integers there corresponds a transformation T of the unit interval into itself; the action of T on a number is induced by the action of π on the indices of the binary expansion of the number. The author proves that T is measure-preserving for all π and that T is ergodic if and only if π has no finite cycles. The major part of the paper studies the convergence of various series; here is a typical result. If f is a real function and if $\{c_n\}$ is a sequence of real constants such that (a) f satisfies a Lipschitz condition of order greater than $\frac{1}{2}$, (b) the mean value of f is 0, and (c) $\sum_n c_n^2 \log^2 n$ converges, then $\sum_n c_n f(T^n x)$ converges almost everywhere. *P. R. Halmos* (Chicago, Ill.).

Jurkat, Wolfgang B. Ein funktionentheoretischer Beweis für O-Taubersätze bei den Verfahren von Borel und Euler-Knopp. Arch. Math. **7** (1956), 278–283.

Let $u_0+u_1+\cdots$ be a series having partial sums $s_0, s_1, \cdots$ and let $B(\sum u_n)=S$ and $(E,q)(\sum u_n)=S$ signify, respectively, that $\sum u_n$ is evaluable to S by the Borel exponential method B and the Euler-Knopp method (E,q). The notation used is that for which the (E,q) transform of $\sum u_n$ is

$$\sigma_n^{(q)}=\frac{1}{q^n}\sum_{k=0}^{n}\binom{n}{k}(q-1)^{n-k}s_k=\sum_{k=0}^{n}\binom{n}{k}r^k(1-r)^{n-k}s_k,$$

where $q=r^{-1}$, so that (E,q) is regular when $q\geqq 1$ and stronger than convergence when $q>1$. This paper gives an exceptionally short and informative proof of the classic theorem: If $B(\sum u_n)=S$ and $n^k u_n=O(1)$, then $\sum u_n$ converges to S. The proof involves three steps. I. If $B(\sum u_n)=S$ and $n^k u_n=O(1)$, then s_n is bounded. II. If $B(\sum u_n)=S$ and s_n is bounded, then $(E,q)(\sum u_n)=S$ for each $q>1$. III. If $(E,q)(\sum u_n)=S$ for each $q>1$ and $n^k u_n=O(1)$, then $\sum u_n$ converges to S. The results in steps I and III are known corollaries of the classic theorem being proved, and short proofs of them are given. The result in step II is not a corollary of the theorem being proved, and it seems to be new and significant. Proof of step II should be the most difficult part of the proof. However the author has found a short function-theoretic proof of more than is required. If $B(\sum u_n)=S$ and $\sum (z^k/k!)s_k=O(e^{|z|})$ as $|z|\to\infty$, then $(E,q)(\sum u_n)=S$ for each $q>1$. *R. P. Agnew* (Ithaca, N.Y.).

See also: Rubel, p. 471; Roelcke, p. 476; Bochner, p. 484; Ogieveckiĭ, p. 541.

Approximations, Orthogonal Functions

Mikeladze, Š. E. Quadrature formulas for a regular function. Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 289–296. (Russian)

For an analytic function $f(z)$ of a complex variable the author gives a general quadrature formula of type

$$(*)\qquad \int_{a_1}^{a+H} f(z)dz=H\sum_{\nu=0}^{n}A_\nu f(a+t_\nu H)+R_n.$$

Assuming (*) is exact ($R_n=0$) for polynomials of degree n, the A_ν may be derived by integrating the Lagrange interpolation polynomial, or by undetermined coefficients. For various fixed points $\{a+t_\nu H\}$ the author derives known quadrature formulas of Salzer [J. Math. Phys. **29** (1950), 96–104; MR **12**, 55], Birkhoff and Young [ibid. **29** (1950), 217–221; MR **12**, 445], and Young [ibid. **31** (1952), 42–44; MR **13**, 782, 1140]. If $f(z)$ is real for real z, the author fixes equal weights A_ν and finds formulas of the Chebyshev type

$$(**)\qquad \int_{-1}^{1}f(x)dx=\frac{2}{n}\sum_{\nu=1}^{n}\operatorname{Re} f(z_\nu),$$

exact for polynomials of degree n. A numerical example with $f(x)=e^{-x^2}$ and $n=10$ yields accuracy to 6 significant decimals.

New quadrature formulas are found with the points $a+ht_\nu$ at the center and vertices of regular polygons. Also, one for $\int_{-H}^{H} f(x)dx$ is given in terms of $f(0)$, $f(\pm H)$, $f(\pm 2H)$, $f(\pm iH)$, and $f(\pm 2iH)$. *G. E. Forsythe.*

Aljančić, S. Über Summierbarkeit von Orthogonalentwicklungen stetiger Funktionen. Acad. Serbe Sci. Publ. Inst. Math. **10** (1956), 121–130.

Let $\{\phi_n\}$ be an orthonormal system in which each function is bounded. The author establishes a necessary and sufficient condition for a row-finite matrix to sum the $\{\phi\}$-expansion of every continuous function to the function. *R. P. Boas, Jr.* (Evanston, Ill.).

Tricomi, Francesco G. Sulla chiusura dei sistemi ortonormali di funzioni. Rev. Un. Mat. Argentina **17** (1955), 299–303 (1956).

The author recalls two convenient necessary and sufficient conditions for the completeness of an orthonormal system $\{\varphi_n(x)\}$ in L^2, given, respectively, by Vitali [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **30** (1921), 2° semestre, 498–501] and Dalzell [J. London Math. Soc. **20** (1945), 87–93; MR **7**, 437], and generalizes these results. If $p(x)$ is a weight-function, L^2 is replaced by $L_p{}^2(a,b)$, $g(x)$ is a suitable function in $L_p{}^2$, and $q(x)$ is any positive function for which the integrals below exist, the author's generalization of Dalzell's condition for the completeness of $\{\varphi_n(x)\}$ in $L_p{}^2$ reads:

$$\int_a^b dy q(y)\int_a^y p(x)g^2(x)dx=\sum_n\int_a^b dy g(y)\left[\int_a^y p(x)g(x)\varphi_n(x)dx\right]^2.$$

(In Dalzell's condition $p(x)=g(x)=q(x)=1$.) The generalized Vitali condition is used to derive the completeness of the system of Laguerre polynomials from a special known expansion. *A. Erdélyi* (Jerusalem).

Motzkin, T. S.; and Walsh, J. L. Least pth power polynomials on a finite point set. Trans. Amer. Math. Soc. **83** (1956), 371–396.

Let $E:(z_1, \cdots, z_m)$ be a set of points of the complex plane, and let $\mu_1, \cdots, \mu_m$ be positive. The first section of this paper is devoted to the explicit determination of polynomials $T_{m-1}(z)=z^{m-1}+\cdots$ minimising

$$(*)\qquad \sum_1^m \mu_k|T_{m-1}(z_k)|^p.$$

Having previously [same Trans. **78** (1955), 67–81; MR **16**, 585] solved the case $p=1$, they treat separately the cases $p>1$ and $0<p<1$, with uniqueness for $p>1$. §§ 2–3 deal mainly with the distribution of the zeros of $T_n(z)$ minimising more general deviations [cf. the previously cited paper and also L. Fejér, Math. Ann. **85** (1922), 41–48]; one such result concerns the separation of E by an arbitrary equilateral hyperbola with the join of two zeros as diameter. § 4 deals with the effect on the $T_n(z)$, for the deviation (*), of adding points to E or removing them; under various conditions the same $T_n(z)$ may be extremal for the modified set, or may be so after the addition or removal of factors. In § 5 the aim is to characterise the set of T_n for deviation (*), where E (now real) is fixed but not the μ_k; for $0<p<1$ and $n<m$ they are the polynomials of the above form which vanish in n points of E; for $p>1$ they are the polynomials whose zeros are simple, lie in the smallest closed interval containing E, and separate strongly a set of $n+1$ points of E on which $T_n\neq 0$, while for $p=1$ the set of T_n can be identified with two other sets. § 6 gives certain consequences of the orthogonality condition appropriate to (*) for $p>1$. Finally § 7 deals with best approximation to a continuous $f(z)$ over a closed bounded set E containing infinitely many points by means of a polynomial $p(z)$ of fixed degree. It is shown that the polynomials $p_k(z)$ corresponding to a suitable sequence of finite point sets E_k converge to $p(z)$, using the deviation

$$\max |\mu(z)(f(z)-p(z))|.$$

The same holds for the deviation corresponding to (*) if $p>1$, while for $0<p<1$ the result is qualified on account of non-uniqueness. *F. V. Atkinson* (Canberra).

See also: Kegel, p. 460; Edrei, p. 471; Langenbah, p. 475; Munschy, p. 476; Thorne, p. 477; Gatteschi, p. 477; Erdös et Karamata, p. 478; Brousse, p. 480; Atkinson, p. 484; Krylov, p. 515; Wintner, p. 518; Lah, p. 520; Wegner, p. 525; Volterra, p. 526; Legendre, p. 530; Glaser, p. 535; Wu, p. 536.

Trigonometric Series and Integrals

Shapiro, Victor L. The uniqueness of double trigonometric series under circular convergence. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 885–887.

If m and n are integers let $a_{m,n}\to 0$ as $m^2+n^2\to\infty$. Theorem: If, as $R\to\infty$, $\sum a_{m,n}e^{i(mx+ny)}\to 0$ for all x and y, the summation being taken for $1\leq m^2+n^2\leq R^2$, then all $a_{m,n}$ vanish. A proof of this is sketched, based on previous results of the author [Proc. Amer. Math. Soc. **4** (1953), 692–695; MR **15**, 418]. *W. W. Rogosinski.*

Hsiang, Fu Cheng. On absolute convergence of Fourier series. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 917–920.

The author gives an elaborate proof that if ϕ is the sum-function of an absolutely convergent Fourier series, then

$$\lim_{x\to 0+} x^{-1}\int_0^x \phi(u)du=\phi(0).$$

However this relation obviously holds for a much wider class of functions, for example, for the continuous functions. *P. T. Bateman* (Princeton, N.J.).

Brousse, P. Séries ultrasphériques et intégrale de Poisson. J. Rational Mech. Anal. **5** (1956), 967–986.

The author defines a "Poisson integral" (called P-integral in this review) whose role in the theory of ultraspherical series is analogous to that of the usual Poisson integral in the theory of trigonometric series. The P-integral satisfies a certain homogeneous partial differential equation and is defined in a semi-circular domain D. The following topics are studied: Behavior of the P-integral near the boundary of D. Dirichlet problem. Form of solutions near a point of discontinuity of the boundary function. Expansion of the P-integral in an ultraspherical series. Poisson summability of ultraspherical series. Behavior of the derivatives of P-integrals near the boundary (analogues of Fatou's theorems). *W. Rudin.*

See also: Postnikov, p. 468; Shapiro, p. 475; Nishimura, p. 478; Helgason, p. 494; Mewes, p. 523; Pickett and Sundara, p. 525; Nikol'skiĭ, p. 525; Uflyand, p. 527; Hoff, p. 538.

Integral Transforms

★ **Churchill, R. V. Extensions of operational mathematics.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 235–250. University of Maryland Book Store, College Park, Md., 1956.

The paper is concerned with certain integral transforms

$$(*)\qquad T\{F(x)\}=\int_a^b K(x,s)F(x)dx$$

that are related to the solution of problems involving differential equations. The major portion of the paper is devoted to the "Sturm-Liouville transform" specified by a non-singular Sturm-Liouville boundary value problem on a finite closed interval $a\leq x\leq b$, in which case (*) is of the form

$$T\{F(x)\}=\int_a^b p(x)\phi_n(x)F(x)dx=f(s_n)\ (n=1,\cdots),$$

where s_n, $\phi_n(x)$ $(n=1,\cdots)$. are the proper values and normalized proper functions of the given boundary value problem, and the transform $f(s_n)$ represents the Fourier coefficients of the function $F(x)$ with respect to the orthonormal set of functions $\phi_n(x)$. The results presented for this case may be described as well-known results for Sturm-Liouville boundary value problems expressed in the language of transforms. In addition there are given a number of examples of special and singular cases, comments are made on transforms which have been shown to possess a "convolution property", and finally there is discussed in a formal fashion a particular type of boundary value problem for partial differential equations in which the method of Sturm-Liouville transforms provides a solution. *W. T. Reid* (Evanston, Ill.).

Bhonsle, B. R. On some results involving generalised Laplace's transforms. Bull. Calcutta Math. Soc. **48** (1956), 55–63.

The author gives several formulas involving combinations of Laplace, Meijer and Whittaker transforms, and obtains as special cases some fairly complicated definite integrals involving hypergeometric functions.

R. P. Boas, Jr. (Evanston, Ill.).

★ **Oberhettinger, Fritz. Tabellen zur Fourier Transformation.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. x+214 pp. DM 35.80.

This volume is the most extensive table of Fourier transforms yet published, containing about 1800 pairs of transforms, many of them new. They are arranged very conveniently, much as in the Bateman tables [Erdélyi et al., Tables of integral transforms, vol. 1, McGraw-Hill, New York, 1954; MR **15**, 868], with the definition of the transform repeated at the top of each page, and no confusing system of abbreviated information. There are long sections of sine and cosine transforms, and about 50 exponential Fourier transforms. All necessary special functions are defined in an appendix. Many, but not all, of the pairs of transforms appear twice: thus we find $(a^2+x^2)^{-1}$ listed under algebraic functions with cosine transform $K_0(ay)$, and $K_0(ax)$ listed under modified Bessel functions with transform $\frac{1}{2}\pi(a^2+y^2)^{-1}$. However, since not all transform pairs do appear twice, an index of formulas involving each special function would have increased the usefulness of the tables. *R. P. Boas, Jr.*

Narain, Roop. Certain properties of generalized Laplace transform. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 311–328.

The transform in question is

$$\Phi(s)=s\int_0^\infty (st)^{m-\frac{1}{2}}e^{-\frac{1}{2}st}W_{k,m}(st)f(t)dt.$$

The author gives some transform pairs and shows that $s^{1-\nu}\Phi(1/s)$ is the transform of $t^\nu\chi_{\nu,k,m}(ty)f(y)$, where $\chi_{\nu,k,m}$ is a certain series of products of Γ-functions and ${}_1F_2$'s. He then introduces

$$2^{\frac{1}{2}}\int_0^\infty (\tfrac{1}{2}ty)^{\nu+\frac{1}{2}}\chi_{\nu,k,m}(\tfrac{1}{4}t^2y^2)f(y)dy$$

as a generalization of the Hankel transform. He discusses the behavior of the kernel and gives some examples.
R. P. Boas, Jr. (Evanston, Ill.).

Griffith, James L. On Weber transforms. J. Proc. Roy. Soc. New South Wales **89** (1955), 232–248 (1956).

The author investigates the L^2 theory of Weber transforms. He also characterizes those functions whose transforms vanish outside a finite interval in a way similar to that used for Hankel transforms in a previous paper [same J. Proc. **89** (1955), 109–115 (1956); MR **17**, 1066]. He gives one convergence theorem which differs somewhat from that given by Watson [Bessel Functions, 2d ed., Cambridge, 1944, pp. 468–469; MR **6**, 64]. {His discussion of $\int_0^\infty$ on p. 247 is inadequate, but he has convinced the reviewer that it can be satisfactorily completed.} *R. P. Boas, Jr.* (Evanston, Ill.).

Stanković, Bogoljub. Inversion d'une transformation intégrale. Acad. Serbe Sci. Publ. Inst. Math. **10** (1956), 85–88.

The author announces an inversion, by means of a differential operator, of the transform

$$F(\sigma)=\int_0^\infty \Phi(0, -\nu; -y\sigma^{-\nu})(\nu y)^{-1}f(y)dy,$$

where

$$\Phi(\mu, -\nu; z)=\sum_{k=0}^\infty \frac{z^k}{\Gamma(k+1)\Gamma(\mu-k\nu)},\quad 0<\nu<1.$$

His method depends on direct use of the differential equation satisfied by the kernel. *R. P. Boas, Jr.*

See also: Hewitt and Zuckerman, p. 465; Bellman, p. 468; Evgrafov, p. 472; Heins, p. 478; Parodi, p. 485; Vaněk, p. ; 488Vasilache, p. 488; Rudin, p. 489; Gelfand et Šilov, p. 493; Wintner, p. 518; Kopzon, p. 526; Uflyand, p. 527; Krzywoblocki, p. 532; Miles, p. 534; Hoff, p. 538; Četaev, p. 545.

Ordinary Differential Equations

Babkin, B. N. On a generalization of a theorem of academician S. A. Čaplygin on a differential inequality. Molotov. Gos. Univ. Uč. Zap. **8**, no. 1 (1953), 3–6. (Russian)

S. A. Čaplygin showed [Byull. Naučno-Eksper. Inst. Inst. Put. Soobšč. **13** (1919), 1–16] that a smooth curve $y=v(x)$ through $A(x_0, y_0)$ is an essential upper integral curve through A of $y'=f(x, y)$ for $x_0\leqq x\leqq x_1$ if

$$v'(x)-f(x, v(x))>0$$

there. The condition is here generalized to

$$v(x)-y_0-\int_{x_0}^x f(t, v(t))dt>0,$$

if f_y is continuous and $\geqq 0$. Similarly for "lower" and "$<$".
F. A. Ficken (Knoxville, Tenn.).

★ **Whyburn, William M. Differential systems with boundary conditions at more than two points.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 1–21. University of Maryland Book Store, College Park, Md., 1956.

This paper is concerned with differential systems consisting of a set of n first order ordinary differential equations in $y_i(x)$, $(i=1, \cdots, n)$, together with boundary conditions that may involve the values of these functions at more than two points of a fixed interval $X: a\leqq x\leqq b$. For systems involving "initial" values at more than one point there is given a brief survey of known results; in the case of a system of the form

$$(*)\qquad y_n'=f_n(x, y),\ y_i'=f_i(x, y)+g_i(x, y)y_{i+1}\qquad (i=1, \cdots, n-1),$$

in $y=(y_1, \cdots, y_n)$ the author employs a theorem of R. Conti [Ann. Mat. Pura Appl. (4) **35** (1953), 155–182; MR **15**, 709] to establish various results concerning the dependence of solutions on the initial points and initial values. There is established also an existence theorem for a system involving a set of differential equations somewhat more general than (*) and initial values at more than one point. The last two sections of the paper are devoted to a survey of the literature on systems consisting of a set of linear differential equations and boundary conditions involving the values of the $y_i(x)$ at the endpoints of X and also at the points of a given subset of $a<x<b$ of suitable character, with special attention to the case of discontinuous solutions restricted by interface conditions. *W. T. Reid* (Evanston, Ill.).

Lefschetz, Solomon. On a theorem of Bendixson. Bol. Soc. Mat. Mexicana (2) **1** (1956), 13–27.

In his memoir on differential equations [Acta Math. **24** (1901), 1–88], Bendixson investigates systems of the form (*) $x'=X(x, y)$, $y'=Y(x, y)$ $('=d/dt)$ where X and Y are real and holomorphic at the origin, $X(0, 0)=Y(0, 0)=0$, and the origin is an isolated critical point. Bendixson

showed that by a finite sequence of quadratic transformations one may replace the study of the critical point at the origin by that of a finite set of ordinary points or isolated critical points where at least one characteristic root is non-zero: the so-called Bendixson systems. It is not easy to obtain the quadratic transformations in any practical instance. In the present paper the author gives a constructive process to reduce the critical point, in a finite number of steps, to ordinary or Bendixson types.

Let n be the lowest degree present in the series for X and Y, and let X_n and Y_n be the homogeneous polynomials of degree n composed of the terms of degree n in these series. Let a TO-curve be a characteristic of (*) tending to the origin in a definite direction. If a TO-curve is tangent to $ax+by=0$, then $ax+by$ is a factor of yX_n-xY_n. The author shows first that if $ax+by$ is not a common factor of X_n and Y_n, so that $ax+by=0$ is not a common tangent to $X=0$ and $Y=0$ at the origin, it is possible to determine quite simply the TO-curves, if any, tangent to $ax+by=0$ at the origin. The cases in which yX_n-xY_n does or does not vanish identically are treated separately.

The case in which $ax+by=0$ is a common tangent to $X=0$ and $Y=0$ at the origin is much more complicated. The author shows that by appropriate changes of variables it is possible to obtain a system with the following properties. X_n and Y_n each contain a term ay^n. When $X=0$ and $Y=0$ are solved for y in terms of x, each has the same numbers of real and complex branches, and each branch is represented with y equal to a series in powers of $x^{1/q}$ for some integer q. Neither axis is a common tangent $X=0$ and $Y=0$, and if $y=mx$ is a common tangent, to $X=0$ and $Y=0$ have the same number of branches tangent to it. If of the branches of $X=0$ tangent to $y=mx$, J have the partial series $mx+\cdots+\sigma x^{(p-1)/q}$ in common, then exactly J branches of $Y=0$ will have these same terms in common. With this assumption, the author gives a complicated but straightforward analysis of the critical directions $y=mx$, covering all possible cases of TO-curves in such a direction.

The author shows that there is always an order for a TO-curve. The order is a number μ such that as $x\to 0$ along the curve, $|y|\,|x|^{-\mu+\varepsilon}\to 0$ and $|y|\,|x|^{-\mu-\varepsilon}\to+\infty$ for any $\varepsilon>0$. This justifies the process used by Briot and Bouquet [Ince, Ordinary differential equations, Longmans-Green, London, 1926, p. 297].

W. S. Loud.

Conti, Roberto. Sulla "equivalenza asintotica" dei sistemi di equazioni differenziali ordinarie. Ann. Mat. Pura Appl. (4) **41** (1956), 95–104.

Using results of Cesari the author generalizes a result of Weyl [Amer. J. Math. **68** (1946), 7–12; MR **7**, 382] by replacing the constant n by n matrix by $A(t)$ where the matrix $dA(t)/dt$ is required to be absolutely integrable on $[0, \infty]$. Further it is assumed that for all $t\geqq 0$, p characteristic roots of $A(t)$ are distinct and have real part zero while the remaining $n-p$ all have real part $\leqq -a^2<0$ for some constant a.

N. Levinson (Cambridge, Mass.).

Vinograd, R. È. On the limit behavior of an unbounded integral curve. Moskov. Gos. Univ. Uč. Zap. 155, Mat. **5** (1952), 94–136. (Russian)

The Poincare-Bendixson theory left unanswered the case of the limit set of an unbounded integral curve. The author gives a very detailed description of the ω-limit set, E_ω, of an integral curve of

$$\frac{dx}{dt}=P(x, y),\quad \frac{dy}{dt}=Q(x, y)$$

which is unbounded for $t\to\infty$. (As is shown, the integral curves of this system, for which continuity and uniqueness of solutions are assumed, may be considered as the trajectories of a dynamical system, i.e., where $t\to\pm\infty$ on each curve.) If E_ω contains no singular points, it consists of at most a countable number of wandering (no α- or ω-limit points) trajectories, each of which separates the plane. When there are no unbounded singular components, the components of E_ω are classified into five types in terms of the behavior of integral curves lying in them. For example: a component B of type II is one which contains a pair K_0, K_0^* of integral curves of E_ω, each of which is unbounded in one direction. Then the α-limit set of one and the ω-limit set of the other are empty, $B-K_0-K_0^*$ is a continuum which is the image under a semi-continuous map of points of the unit interval into singular components or ordinary points in a manner introduced by Solncev [Izv. Akad. Nauk SSSR. Ser. Mat. **9** (1945), 233–240; MR **7**, 117]. (Misprints and a very involved argument make it difficult for the reviewer to follow the details of case V.) Examples are given to show, for instance, that 1) there may exist a countable number of distinct curves in E_ω, even in the absence of singular points; 2) when there are unbounded singular components, E_ω may contain an uncountable number of components, none of which separate the plane. These constructions are facilitated by the theorem that a plane set may be the ω-limit set of a trajectory of some dynamical system if and only if it is the boundary of a simply connected domain.

L. W. Green (Zürich).

Virgopia, Nicola. Su una generalizzazione dell'equazione differenziale di T. Uno ed R. Yokomi. Rend. Mat. e Appl. (5) **15** (1956), 153–176.

The author studies the solutions of the equation

$$\frac{dy}{dx}=\frac{-xy}{P(y^2)-(x+1)[(x-1)^2+\lambda]},$$

where x, y, and the parameter λ are real, and $P(y^2)$ is a function of class C^2 such that $P(0)=0$. The chief question under consideration is that of the existence or non-existence of solutions represented by closed curves in the xy-plane. If $\lambda\leqq 0$ or $\lambda\geqq 1$, no such solutions exist. If $0<\lambda<1$ and $P(y^2)$ is monotone increasing for $y\geqq 0$, there exist at least two such solutions. If $0<\lambda<1$ and $P(y^2)$ has just one maximum in the range $y\geqq 0$, this maximum being positive, there exist such solutions provided a certain subsidiary condition is satisfied. The methods used are similar to those used by Sansone and Conti in discussing the case $P(y^2)\equiv y^2$ [Ann. Mat. Pura Appl. (4) **37** (1954), 37–59; MR **16**, 478].

L. A. MacColl.

Sansone, G.; e Conti, R. Soluzioni periodiche dell'equazione $\ddot{x}+f(x)\dot{x}+g(x)=0$ avente due soluzioni singolari. Abh. Math. Sem. Univ. Hamburg **20** (1956), 186–195.

Studies of pericdic solutions of equations of the form $\ddot{x}+f(x)\dot{x}+g(x)=0$ usually involve the assumption that $xg(x)>0$ for all $x\neq 0$. In this paper the authors investigtea the existence of periodic solutions in certain cases in which this assumption does not hold. A complicated set of seven conditions is found, under which the family of trajectories in the $x\dot{x}$-plane has two singular points, an

unstable node or focus and a saddle-point, and under which there exists at least one periodic solution. Conditions are also found which insure the existence of a unique periodic solution, and some variants of the main results are indicated. *L. A. MacColl.*

Grabar', M. I. On the spectrum of harmonized dynamical systems. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 687–689. (Russian)

Beesack, Paul R. Nonoscillation and disconjugacy in the complex domain. Trans. Amer. Math. Soc. **81** (1956), 211–242.

The equation $w''(z)+p(z)w(z)=0$, with $p(z)$ analytic in a domain D, is called disconjugate on D if no solution $w\not\equiv 0$ vanishes more than once in D, non-oscillatory on D if no solution vanishes infinitely often in D. Criteria for disconjugacy or non-oscillation are given; mostly they involve properties of $M(r)=\sup_{|z|=r}|p(z)|$, D being a disk $|z|<R$. Typical results: (A) Disconjugacy holds in $|z|<b\leq 1$ if $(1-r^2)^2M(r)$ is nonincreasing for $0\leq r<b$ and if $y''+M(r)y=0$ has a solution which does not vanish for $-b<r<b$. (B) If $(1-r^2)M(r)\leq n(n+1)$, where n is an odd positive integer, and if b is the least positive zero of $P_n'(r)$, then there is disconjugacy in $|z|<b$. (C) If $y''+M(t)y=0$ has a solution with $y(0)=0$ and $y'(t)\neq 0$ for $0\leq t<R$, then there is disconjugacy in $|z|<2^{-1}R$. (D) If $y''+M(t)y=0$ has a solution with $y(t)\neq 0$ for $0<t<R$, then there is disconjugacy in $|z|<5^{-1}R$. (E) If D lies in the half-strip $x\geq 0$, $|y|\leq 1$, and includes the real axis $x\geq 0$, if $|p(x+iy)|\leq \delta(x)$ where $\delta(x)\to 0$ as $x\to+\infty$ and $\int_0^\infty x|p(x)|dx<\infty$, then there is non-oscillation in D. (A) includes as special cases several results proved by Z. Nehari [Bull. Amer. Math. Soc. **55** (1949), 545–551; MR **10**, 696] or announced by V. V. Pokornyĭ [Dokl. Akad. Nauk SSSR (N.S.) **79** (1951), 743–746; MR **13**, 222]; (C) and (D) were also announced by Pokornyĭ; (E) and its method of proof are closely related to work of Z. Nehari [Amer. J. Math. **76** (1954), 689–697; MR **16**, 131]. Many other results are proved, some of them being extensions or refinements of known results. *G. E. H. Reuter* (Manchester).

Bruwier, L. Sur quelques applications d'un opérateur différentiel. Mathesis **65** (1956), 337–355.

The author considers polynomial operators $F(D)$ (variable coefficients) and $f(D)$ (constant coefficients), here $(D+x)^r=\sum_{j=0}^r C_r^j x^j D^{r-j}$. It is shown that the equation $f(D+x)y=0$ can be transformed by $y=z\exp(-x^2/2)$ to a linear differential equation in z with constant coefficients, and conversely; e.g., the equations $(D+x)^n y=0$ and $H_n(D)z=0$, where H_n is the Hermite polynomial of degree n, are so related.

Let $F^*(D+x)$ be obtained by replacing powers of $D+x$ by successive iterations of $D+x$ in $F(D+x)$. The equation $F^*(D+x)$ is related to $F(D)z=0$ by the above substitution. A general integral of $f^*(D+x)y=0$ is obtained in terms of the roots of $f(x)=0$, if the roots are distinct; thus a general integral of $(D+x)^n y=0$ is shown to be expressible in terms of the roots of $H_n(x)=0$.

A known formula for $F(D)\{vu\}$ in terms of derivatives of v and $F(D)$ yields integrals of an equation in u whose coefficients are essentially derivatives of a known function. F and the known function are specialized in various ways to obtain different equations; in particular, equations whose coefficients are certain kinds of polynomials (e.g. Hermite polynomials) are obtained.

Some other results are obtained, and several examples are given. *W. J. Coles* (Salt Lake City, Utah).

Wintner, Aurel. Bounded matrices and linear differential equations. Amer. J. Math. **79** (1957), 139–151.

Ordinary, homogeneous, linear differential equations in Hilbert space are considered. The first part of the paper discusses existence, uniqueness, boundedness, and non-singularity for matrices of solutions of the differential systems. Later sections treat the asymptotic behavior of solution vectors as the independent variable, t, becomes infinite. Basic hypotheses in these asymptotic considerations include continuity and boundedness of the coefficient matrix for each fixed t (but not uniformly as t becomes infinite). Under suitable additional hypotheses, the length of each non-identically vanishing solution (vector) is shown to approach a finite, non-zero limit as t becomes infinite. Through adaptation of the Lagrange device of variation of constants, consideration is given to solution changes produced by "small perturbations" in the system.

An appendix contains the conclusions: (1) In order that a bounded matrix be logarithmizable, it is (necessary but) not sufficient that it be non-singular; (2) the logarithmizable matrices fail to form a group. *W. M. Whyburn.*

Conti, Roberto. Sulla t-similitudine tra matrici e la stabilità dei sistemi differenziali lineari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 247–250.

The linear systems (A): $\dot{y}=A(t)y$ and (B): $\dot{z}=B(t)z$ are said to be t-similar if there exists a transformation $y=T(t)z$ which takes (A) into (B), where $T(t)$ is absolutely continuous and both $T(t)$ and $T^{-1}(t)$ are bounded for $t\geq 0$. It is shown that t-similarity preserves various types of stability. The two systems are said to be t_∞-similar if (A) can be transformed, by $y=T(t)z$, into $\dot{z}=(B(t)+C(t))z$ with $\int_0^\infty \|C(t)\|dt<\infty$. It is shown that t_∞-similarity preserves some kinds of stability (but not Liapounov stability). The class of systems considered is wider than that considered by L. Markus [Math. Z. **62** (1955), 310–319; MR **17**, 37]: $A(t)$ and $B(t)$ are merely supposed measurable and summable over every finite interval $0\leq t\leq t_0$. *G. E. H. Reuter* (Manchester).

Hahn, Wolfgang. Eine Bemerkung zur zweiten Methode von Ljapunov. Math. Nachr. **14** (1955), 349–354 (1956).

Consider the differential equation (1) $x'=f(x)$, where x and $f(x)$ are n-dimensional vectors and $f(0)=0$. The second method of Ljapunov for studying the stability of the solution $x=0$ of (1) consists of finding a (Ljapunov) function $V(x)$ such that $V(x)$ and

$$V'(x)=\sum_{j=1}^n (\partial V/\partial x_j)x_j'=\sum_{j=1}^n (\partial V/\partial x_j)f_j(x)$$

satisfy certain properties. In particular, (i) if V is positive definite and V' is negative semidefinite, $x=0$ is stable; (ii) if V is positive definite and V' is negative definite, $x=0$ is asymptotically stable; (iii) if V is either indefinite or positive semidefinite and V' is positive definite, $x=0$ is unstable. The author shows that for $f(x)=Ax$, where A is an $n\times n$ matrix, one can find a Ljapunov function using only matrix theory. The stability problem is then easily discussed using this function. *J. K. Hale.*

Seifert, George. Limiting sets of trajectories of a pendulum-type system. Proc. Amer. Math. Soc. **7** (1956), 1082–1084.

Consider the equation (1) $\theta'=z$, $z'=G(\theta,z)$, where

$G(\theta, z)$ is continuous in (θ, z) and satisfies some condition which insures the uniqueness of the solution of (1) at each ordinary point. Also, suppose $G(\theta+2\pi, z)=G(\theta, z)$ for all θ and that $G(\theta, 0)=0$ has at most a finite number of roots on $0\leq\theta<2\pi$. Those closed curves on the phase cylinder which surround the cylinder are said to be of L_2 type and those closed curves which do not surround the cylinder are of ordinary type. Theorem. If

$$\lim_{|z|\to\infty} G(\theta, z)/z=H(\theta)$$

uniformly for all θ and if $\int_0^{2\pi} H(\theta)d\theta<0$, then the positive limiting set of the trajectory of each solution of (1) is contained in a finite part of the phase cylinder and consists of one of the following configurations: (i) a critical point of (1), (ii) a closed curve of L_2 type which may be an L_2 cycle, (iii) a closed curve of ordinary type which may be an ordinary cycle. Stronger results concerning the nature of these finite limiting sets have been obtained by the author [Z. Angew. Math. Phys. **7** (1956), 238–247; MR **17**, 1207] for the special case where $G(\theta, z)=g(\theta)-f(\theta, \alpha)z$.

J. K. Hale (St. Paul, Minn.).

Laitoch, Miroslav. Sur une théorie des critères comparatifs sur l'oscillation des intégrales de l'équation différentielle $u''=P(x)u$. Publ. Fac. Sci. Univ. Masaryk **1955**, 255–266. (Czech and Russian summaries)

The author unifies portions of oscillation theory for solutions of differential equations $y''+p(x)y=0$ by noting that if $y(x)$ is a solution of the differential equation $y''=Q(x)y$, then $z(x)=[\alpha'(x)]^{-\frac{1}{2}}y[\alpha(x)]$ is a solution of the differential equation $z''=q(x)z$, where

$$q(x)=Q[\alpha(x)]\alpha'^2(x)+[\alpha'(x)]^{1/2}[(\alpha'(x))^{-1/2}]''.$$

W. Leighton (Pittsburgh, Pa.).

Atkinson, C. P. Electronic analog computer solutions of nonlinear vibratory systems of two degrees of freedom. J. Appl. Mech. **23** (1956), 629–634.

This paper is a study of the oscillation of a coupled vibrating system with two degrees of freedom. It may be realized as two masses connected by two springs. In this investigation, one spring is linear, the other nonlinear with cubic nonlinearity. Both the hardening and softening cases are studied for both free and forced oscillations.

The differential equations are solved on an analog computer, and the results compared with approximate solutions obtained by the Ritz approximation method. The results are presented graphically as conventional amplitude-frequency plots. The author points out the usefulness of analog computers for regions in which the Ritz approximation is inaccurate. The author proposes a "suspected stability criterion" on the basis of the computer results. He infers the instability of a solution predicted by analytical methods which can not be obtained with the analog computer. *W. S. Loud.*

Burštein, È. L.; and Solov'ev, L. S. On the theory of strong focussing. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 721–724. (Russian)

The authors consider the differential equation

$$\ddot{x}+\varepsilon p(\varepsilon t, \theta)\dot{x}+q(\varepsilon t, \theta)x=0$$

where ε is a small parameter; p and q are periodic in θ; $\dot{\theta}=\nu(\varepsilon t)$. Under these assumptions, p and q may be regarded as approximately periodic functions in t with slowly varying amplitudes and period. Let x_n and $\dot{x}_n$ denote the values of x and $\dot{x}$, respectively, at the end of the nth period. The determination of the x_n and $\dot{x}_n$ is reduced to the solution of a difference equation with slowly varying coefficients. A discrete analog of the W.K.B. method is used to solve this equation. *L. A. Zadeh.*

Reissig, Rolf. Neue Methoden der nichtlinearen Mechanik von Krylow und Bogoljubow. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **5** (1955/56), 99–102. (Russian, English and French summaries)

Two methods for obtaining approximate solutions of the equation $\ddot{x}+\omega^2x+\varepsilon f(x, \dot{x})=0$ are sketched. One is to assume a solution of the form $x=a\sin(\omega t+\varphi)$, where a and φ are slowly varying functions of t. The other is to seek a solution in the form of a power series in ε, the coefficients of which are periodic functions. *R. Ellis.*

Wintner, Aurel. On an instability criterion of Liapounoff. Portugal. Math. **15** (1956), 13–18.

The author discusses the system of equations

$$(m_1x_1'', \cdots, m_nx_n'')=\text{grad } U(x),$$

where $U(x)$ is a twice continuously differentiable function on a sphere D of radius a in n-dimensional x-space. He proves the following assertions of Painlevé [C. R. Acad. Sci. Paris **125** (1897), 1021–1024]: If h is the energy constant, then (i) $h>0$ implies the solution path $x(t)$ must reach the boundary of D at some finite t no matter how small the radius a of D, (ii) $h=0$ implies either $x(t)$ must behave as in (i) or else $x(t)$ is an unrestricted solution satisfying (2) $x(t)\to 0$ as $t\to\infty$. By means of an example, he shows the following assertion of Painlevé is not true: $h<0$ implies for every $x(t_0)$ it is possible to choose $x'(t_0)$ in such a way that (2) will hold for the solution $x(t)$ determined by $x(t_0)$, $x'(t_0)$ and belonging to the given value of h.

J. K. Hale (St. Paul, Minn.).

Blaquière, Augustin. Equation de Hill non linéaire et méthode stroboscopique de N. Minorsky. C. R. Acad. Sci. Paris **243** (1956), 1711–1714.

The author uses Minorsky's stroboscopic method to examine the equation

$$\frac{d^2x}{d\theta^2}+[Q_0^2-k\delta(\theta)]x-\beta x^3=0,$$

which arises in the design of the strong focussing cosmotron. *A. S. Householder* (Madison, Wis.).

★ **Bochner, S. Sturm-Liouville and heat equations whose eigenfunctions are ultraspherical polynomials or associated Bessel functions.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 23–48. University of Maryland Book Store, College Park, Md., 1956.

For fixed $\gamma\geq 0$ let $P_n(x)=P_n^{(\gamma)}(x)$ $(n=0, 1, \cdots)$, be the polynomial solution of $(1-x^2)P''-(2\gamma+1)xP'=-\lambda_nP$ where $\lambda_n=n(n+2\gamma)$, normalised by $P_n(1)=1$, and write $\rho_n=1/\int_{-1}^{1}\{P_n(x)\}^2(1-x^2)^{\gamma-\frac{1}{2}}dx$. For $f(x)$ $(-1\leq x\leq 1)$ with absolutely convergent Fourier expansion $\sum\rho_na_nP_n(x)$ consider the mapping $f(x)\to f(x; y)$ given by

$$f(x; y)=\sum\rho_na_nP_n(x)P_n(y).$$

The first part of the investigation centers round the fact that this mapping has "property P", namely that $f(x)\geq 0$ in $-1\leq x\leq 1$ implies $f(x; y)\geq 0$ in $-1\leq x, y\leq 1$. There is a continuous analogue of this result in terms of, in effect, Bessel functions. Apparently distinct examples of property P can be constructed from functions on a group in

terms of group-representations. In the next section a study of the partial differential equations

$$(1-x^2)f_{xx}-(2\gamma+1)xf_x=f_t$$

leads to the kernel $K(x, y; t)=\sum \rho_n \exp(-t\lambda_n)P_n(x)P_n(y)$. The fact that $K(x, y; t)\geqq 0$ is deduced in a rather striking way from a study of the semi-ring of bounded sequences $\{c_n\}$ such that $\sum \rho_n a_n P_n(x)\geqq 0$ in $-1\leqq x\leqq 1$ (postulating absolute convergence) implies $\sum \rho_n a_n c_n P_n(x)\geqq 0$ in $-1\leqq x\leqq 1$. General expressions are found for such $\{c_n\}$, and also for a "homogeneous stochastic process" $\{c_n(t)\}$ [for definitions and formulae see the review of the author's previous note, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 1141–1147; MR **16**, 834]. Continuous analogues, corresponding to $f_{xx}+(2\gamma/x)f_x=f_t$, are also formulated. Finally, in connection with the probabilistic interpretation, the author outlines the role of "subordinators" in assessing the relative primacy of homogeneous processes, as explained more fully in his book "Harmonic analysis and the theory of probability" [Univ. of California Press, 1955, Ch. 4; MR **17**, 273]. *F. V. Atkinson.*

Lidskiĭ, V. B. On the completeness of the system of eigen- and associated functions of a non-self-adjoint differential operator. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 172–175. (Russian)

The main theorem of the paper is as follows: If the functions $q(x)\geqq 0$, and $r(x)$ are locally L_1 for $-\infty<x<\infty$, and if with positive constants α and c the following conditions are satisfied, i) $\liminf_{|x|\to\infty} q(x)/|x|^\alpha\geqq c$, ii) $\lim_{|x|\to\infty} |r(x)|/q(x)=0$, then the non-self-adjoint operator $-y''+(q+r)y$ considered on its maximal domain in $L_2(-\infty, +\infty)$ has a discrete spectrum and the eigenfunctions together with the associated functions are complete in L_2.

The advance made by the author compared to previous results in the subject is in weakening conditions on q and r as well as doing away with a priori restrictions on the distribution of the eigenvalues [see, e.g., M. A. Naĭmark, same Dokl. (N.S.) **98** (1954), 727–730; MR **16**, 1032; and J. Schwartz, Pacific J. Math. **4** (1954), 415–458; MR **16**, 144]. An essential tool in the proof is a result of M. V. Keldyš [Dokl. Akad. Nauk SSSR (N.S.) **77** (1951), 11–14; MR **12**, 835]. *N. Aronszajn* (Lawrence, Kan.).

Hull, T. E.; and Julius, R. S. Enclosed quantum mechanical systems. Canad. J. Phys. **34** (1956), 914–919.

Given the differential equation and boundary conditions

$$u''+r(x)u+\lambda u=0,\ u(0)=u(b)=0,$$

what is the value of λ for the same equation but with boundary conditions $u(0)=u(a)=0$, $a<b$? A general formula is developed for the change in λ when a approaches b. The result is evaluated for a number of cases. *H. Feshbach* (Cambridge, Mass.).

Parodi, Maurice. Sur une propriété de certaines équations différentielles d'Euler. Application à la formation de combinaisons linéaires de n fonctions $f_i(t)$ qui admettent pour transformée de Laplace une combinaison linéaire des n fonctions $f_i(s)$. C. R. Acad. Sci. Paris **243** (1956), 1991–1993.

The author shows that under certain conditions the Euler differential equation

$$t^n y^{(n)}(t)+a_1 t^{n-1}y^{(n-1)}(t)+\cdots+a_n y(t)=0$$

is its own Laplace transform. *R. P. Boas, Jr.*

See also: Seidenberg, p. 463; Schubart, p. 472; Kuranishi, p. 474; Churchill, p. 480; Inaba, p. 493; Bielecki, p. 494; Ergin, p. 515; Blaquière, p. 523; Quilghini, p. 528; Glaser, p. 535; Pussèt, p. 536; Savinov, p. 536; Popov, p. 549.

Partial Differential Equations

Łojasiewicz, S. Sur le problème de Cauchy pour les systèmes d'équations aux dérivées partielles du premier ordre dans le cas hyperbolique de deux variables indépendantes. Ann. Polon. Math. **3** (1956), 87–117.

The author gives a method to construct a solution $z_i=z_i(x, t)$, $i=1, \cdots, n$, of the Cauchy problem

$$(*)\quad \frac{\partial z_i}{\partial t}=f_i(x, t, z_1, \cdots, z_n, q_1, \cdots, q_n),\ z_i(x, 0)=z_{i0}(x),$$

under weak continuity conditions on the f_i and the initial data $z_{i0}(x)$. Here $q_j=\partial z_j/\partial x$, and the system (*) is assumed to be hyperbolic in the sense that the characteristic roots of the matrix $[\partial f_i/\partial q_j]$ are all real and distinct. The continuity conditions assumed here parallel those used by R. Conti [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12** (1952), 61–65, 151–155; MR **14**, 172; cf. also A. Douglis, Comm. Pure Appl. Math. **5** (1952), 119–154; MR **14**, 655] who treated a system in which the f_i did not depend on the q_k for $k\neq i$. While a complete listing of all the restrictions would take too much space, we give here the essential details of the main result. Let z_{i0} and f_i be periodic in x and have first partial derivatives which satisfy Lipschitz conditions. Assume the characteristic roots of the matrix $[\partial f_i/\partial q_j]$ are all real, distinct, and positive. Then there exists a region $0\leqq t\leqq\delta$, $-\infty<x<\infty$ in which (*) has a unique solution in the class of functions whose first derivatives satisfy Lipschitz conditions. If it is only a question of existence of a solution, the conditions of periodicity on z_{i0} and f_i, as well as the positiveness of the characteristic roots, can be dropped. Conditions under which the solution exists for $0\leqq t<\infty$ are also given. *F. G. Dressel* (Durham, N.C.).

Bulešev, U. B. Extension of the method of Lagrange-Charpit to the case of two equations of the first order with two unknown functions. Akad. Nauk Uzbek. SSR. Trudy. Inst. Mat. Meh. **16** (1955), 39–44. (Russian)

To obtain the said extension the author adjoins to a given system of differential equations

$$F_i(x, y, p_1, p_2, q_1, q_2)=0\ (i=1, 2),$$

where p_1, p_2, q_1, q_2 are partial derivatives of z_1 and z_2, respectively, two equations $\phi_i(x, y, z_1, z_2, p_1, p_2, q_1, q_2)=0$ such that these together with $F_i=0$ can be solved for p_1, p_2, q_1, q_2 and, furthermore, after substitution of these solutions in

$$-dz_1+p_1dx+q_1dy=0 \text{ and } -dz_2+p_2dx+q_2dy=0$$

one obtains exact differentials. The author takes the necessary and sufficient conditions for integrability expressed by outer differential forms (Frobenius' theorem) and expands these forms. He obtains a system of equations whose coefficients are outer differential quadratic forms. The practical utility of this form of equations is highly questionable because of its complexity. To reproduce his system of equations in a compact form let **A** and **B** be

skew-symmetric dyadics in 8-dimensional space

$$\{(x, y, z_1, z_2, p_1, p_2, q_1, q_2)\}$$

and ∇ be the gradient operator in this space; then ϕ_1 must satisfy $\nabla\phi_1 \cdot \mathbf{A} \cdot \nabla\phi_2 = 0$ and $\nabla\phi_2 \cdot \mathbf{B} \cdot \nabla\phi_1 = 0$. Components of $\mathbf{A}$ and $\mathbf{B}$ are quadratic or cubic forms in partial derivatives of F_1. From this it is apparent that determination of ϕ_1 is not an easy matter. *C. Masaitis.*

Volpato, Mario. Sull'espressione differenziale:

$$p(x, y)dx + q(x, y)dy$$

nell'ambito delle funzioni misurabili rispetto ad una e continue rispetto all'altra variabile. Rend. Sem. Mat. Univ. Padova **25** (1956), 303–306.

In elementary analysis, the expression

$$p(x, y)dx + q(x, y)dy$$

is termed the total differential of the function $F(x, y)$ in the rectangle $R: a \leq x \leq b$, $c \leq y \leq d$ if $F_x = p$, $F_y = q$ in R. It is then evident that the validity of the relation

$$\int_a^x [p(u, y) - p(u, c)]du = \int_c^y [q(x, v) - q(a, v)]dv, \quad (x, y) \in R,$$

is necessary and sufficient for an expression

$$p(x, y)dx + q(x, y)dy$$

to be a total differential. The purpose of this note is to call attention to several non-elementary situations where the above integral relation may be used to characterize total differentials. The statements in the note follow readily from previous results in the literature. *T. Radó.*

Haimovici, M. Sur les prolongements partiels d'un système de Pfaff du II-ème genre. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), 91–104. (Romanian. Russian and French summaries)

Haimovici, M. Sur le prolongement des équations du II-e ordre à une fonction inconnue de deux variables indépendantes et sur les transformations de ces équations. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **1** (1955), 69–136. (Romanian. Russian and French summaries)

Greco, Donato. Nuove formole integrali di maggiorazione per le soluzioni di un'equazione lineare di tipo ellittico ed applicazioni alla teoria del potenziale. Ricerche Mat. **5** (1956), 126–149.

Let $\mathfrak{M}$ be the elliptic differential operator defined by

$$\mathfrak{M} = \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x) \frac{\partial}{\partial x_i} + c(x).$$

Here x denotes the point with coordinates $(x_1, \cdots, x_m)$ of a domain T in an m-dimensional (real) Euclidean space S_m. T is supposed to be of class $A^{(2)}$, i.e. (essentially) its boundary consists of a finite number of $(m-1)$-dimensional twice continuously differentiable manifolds. Concerning the coefficients the following assumptions are made in T: the $a_{ij}(x)$ belong to the class $C^{(0,\lambda)}$ $(0 < \lambda < 1)$ and the $b_i(x)$, $c(x)$ belong to the $C^{(0)}$. ($C^{(p)}$ is the class of functions with continuous pth derivatives, and $C^{(p,\lambda)}$ is the subclass of $C^{(p)}$ of functions whose pth derivatives satisfy a Hölder condition of index λ).

The main results of the paper are then as follows: I. Let u be a solution of class $C^{(2,\lambda)}$ of the equation

$$\mathfrak{M}(u) = f(x), \quad f(x) \in L^p(T). \tag{2}$$

Let Δ be a domain situated in the interior of T. Then the following a priori estimate holds:

$$\int_\Delta \sum_{i,j=1}^{m} \left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|^p dx \leq C_p \int_T \left(|f(x)|^p + |u(x)|^p + \sum_{i=1}^{m} \left|\frac{\partial u}{\partial x_i}\right|^p\right) dx, \tag{3}$$

where C_p depends only on the following quantities: i) the distance from Δ to the boundary of T; ii) the maximum of the absolute values of the coefficients of $\mathfrak{M}$; iii) the modulus of continuity of the $a_{ij}(x)$; iv) the minimum of the absolute value of the determinant of the $a_{ij}(x)$.

II. This result concerns an a priori estimate in the whole domain T; it is shown that (3) still holds with the following modifications: the integration in its left member is extended over T instead of Δ and in its right member the product of C_p is multiplied by the integral over the boundary of T of the sum of the absolute values of the pth powers of u and all its first and second derivatives. Except for the domain T, C_p depends only on the quantities ii)–iv) of I.

III. This result concerns "generalized potentials" and their derivatives: let $L(x, y)$ be a Levi function corresponding to the operator $\mathfrak{M}$ (i.e. essentially a function having the "right" singularity at $x = y$ but not necessarily satisfying $\mathfrak{M}L = 0$; for the exact definition see e.g. C. Miranda, Equazioni Alle Derivate Partiale, Di Tipo Ellittico, Springer 1955 (p. 13)). Then a generalized potential W is defined

$$W(x) = \int_T L(x, y)z(y)dy. \tag{4}$$

Under the assumption that $a_{ij} \in C^{(0,\lambda)}$ and $z \in L^p$ it is then proved: if $p \geq 1$, then $W(x)$ is absolutely continuous in x_i for almost all $(m-1)$-tuples $(x_1 \cdots x_{i-1}, x_{i+1} \cdots x_n)$; and the $\partial W/\partial x_i$ (which therefore exist almost everywhere) are in L^p. If $p > 1$, the second derivatives $\partial^2 W/\partial x_i \partial x_j$ exist almost everywhere, are in L^p, satisfy an inequality of the form

$$\int_T \sum \left|\frac{\partial^2 W}{\partial x_i \partial x_j}\right|^p dx \leq C_p \int_T |z(x)|^p dx \tag{5}$$

and the integral equation

$$W(x) = -z(x) + \int_T \mathfrak{M}_x L(x, y)z(y)dy, \tag{6}$$

where $\mathfrak{M}_x$ denotes $\mathfrak{M}$ operating on the variable x. Thus W is a solution of the Poisson equation (2) (with $f = -z$) if the Levi function $L(x, y)$ is an elementary solution, i.e. satisfies $\mathfrak{M}_x L(xy) = 0$. Finally, W satisfies a Hölder condition if $p > \frac{1}{2}m$, and has first derivatives satisfying a Hölder condition if $p > m$.

The author makes the following historical remarks (among others): The results I and II can be considered as an extension of results obtained by Caccioppoli [Giorn. Mat. Battaglini (4) **480**(1951), 186–212; MR 13, 749] in the case $p = 2$. III is an extension of results of Lichtenstein [J. Reine Angew. Math. **141** (1912), 12–42] and Friedrichs [Duke Math. J. **14** (1947), 67–82; MR 8, 583] concerning the case $p = 2$ of Newtonian potentials (Lichtenstein treats the logarithmic potential, i.e. the case $m = 2$, while Friedrichs treats arbitrary m). Corresponding results for generalized potentials, still for $p = 2$, can be found in Miranda's book quoted above. Essential use is made in the present paper of results obtained by Calderon and Zygmund [Acta Math. **88** (1952), 85–139; MR 14, 637] for arbitrary p in the case of the Newtonian potential. *E. H. Rothe.*

Greco, Donato. Un teorema di esistenza per il problema di Dirichlet relativo ad un'equazione lineare ellittica in m variabili. Ricerche Mat. **5** (1956), 150–158.

Using a refinement of the a priori estimate described as result "II" in the preceding review the author proves an existence theorem which with the notations of the review referred to reads as follows: let $y(x)$ be a function in $C^{(2)}$ on the boundary of T. The latter is of class $A^{(2,\lambda)}$ and the coefficients of the eq. (2) of the preceding review satisfy the following assumptions in T: the a_{ij} are continuous, while b_i, c, f are measurable and bounded $c \leqq 0$. Then the Dirichlet problem given by (2) and φ has at least one "solution" u of class $C^{(1,\lambda)}$: u has an absolutely continuous derivative with respect to x_i for almost all $(m-1)$-tuples $(x_1 \cdots x_{i-1}, x_{i+1} \cdots x_m)$; the second derivatives exist almost everywhere in T and u satisfies (2) almost everywhere in T. The idea of proof is to approximate the equation (2) by a sequence of equations whose coefficients are regular enough to insure a unique solution in $C^{(2\lambda)}$ of the corresponding boundary value problem. The a priori limitation is then used to prove the compactness of the sequences of solutions thus obtained. *E. H. Rothe.*

Colombo, Serge. Sur les solutions de l'équation de Laplace dans le cas d'une symétrie cylindrique. C. R. Acad. Sci. Paris **243** (1956), 1471–1473.

The solution $u(r, z)$ of

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

under the conditions

$$u(r, 0) = f(r),\ u(r, +\infty) = 0$$

is obtained in the form

$$u(r, z) = \frac{1}{2\pi i}\int_{c-\omega i}^{c+\omega i} e^{tr^2 p}\Phi\left(\frac{1}{p}\right)\frac{dp}{p},$$

where

$$\Phi(p) = \int_0^\infty e^{-ps - z\sqrt{(2s)}} f(\sqrt{2s})\,ds,$$

$$f(\xi) = \int_0^\infty f(\rho) J_0(\rho\xi)\rho\,d\rho.$$

E. T. Copson (St. Andrews).

Višik, M. I. On the first boundary problem for elliptic equations in a new functional aspect. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 781–784. (Russian)

En utilisant des fonctions poids convenables, l'A. résout le problème de Dirichlet (dans un sens "faible") par des méthodes hilbertiennes voisines de la méthode des projections, pour des fonctions qui ne sont pas à Dirichletien fini, et pour des seconds membres qui sont des distributions plus générales que des dérivées du premier ordre (au sens des distributions) de fonctions de carré sommable. L'A. généralise la méthode aux opérateurs elliptiques d'ordre quelconque (mais ne donne alors aucune indication sur le choix des fonctions poids); pour l'ordre 2, l'A. utilise (pour les fonctions poids) O. A. Oleĭnik, Mat. Sb. N.S. **30(72)** (1952), 695–702 [MR **14**, 280] et pour les propriétés des espaces fonctionnels introduits, M. I. Višik, ibid. **35(77)** (1954), 513–568 [MR **16**, 927]. *J. L. Lions.*

Ásgeirsson, Leifur. Some hints on Huygens' principle and Hadamard's conjecture. Comm. Pure Appl. Math. **9** (1956), 307–326.

This is a reprint of a talk given by Professor Ásgeirsson at the Symposium on Partial Differential Equations held at Berkeley in June 20–July 1, 1955. It gives an extremely interesting account of his unpublished researches during the last twenty years on Huygens' Principle and on the conjecture of Hadamard that Huygens' Principle holds for a hyperbolic equation if and only if the equation is "equivalent" to the wave-equation with three spatial dimensions.

The work is closely related to Beltrami's proof of Poisson's solution of the initial value problem for the wave-equation and the author's own use of spherical mean values in the theory of the ultrahyperbolic equation. The starting point is the fact that, for the wave-operator

$$\Delta_{2m-1} - \frac{\partial^2}{\partial t^2}$$

in space of $2m-1$ dimensions, there is a differential operator R of order $m-2$, such that

$$R\left(\Delta_{2m-1} - \frac{\partial^2}{\partial t^2}\right)u$$

is equal to the divergence of a vector on the characteristic cone. It is impossible to give more details in a brief space.

E. T. Copson (St. Andrews).

Douglis, Avron. A criterion for the validity of Huygens' principle. Comm. Pure Appl. Math. **9** (1956), 391–402.

In this paper is supplied a proof of an analytical criterion for the validity of Huygens' Principle among linear hyperbolic equations of the second order in four independent valuables. The argument is based on a new procedure of obtaining an integral fromula for the solution of Cauchy's problem; this procedure comes out of an integral equation theory of hyperbolic equations, developed by the author in a previous paper. [same Comm. **7** (1954), 271–295; MR **16**, 44]. *E. T. Copson.*

Galonen, L. M. On a certain simplification of a method of finding functionally invariant solutions of the wave equation. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. **32** (1955), no. 4, 173–178. (Russian)

A functionally invariant solution of the wave equation is a function u such that an "arbitrary" function of it, $f(u)$, is a solution of the wave equation. For two space variables, x, y, the determination of all functionally invariant solutions of the wave equation $u_{xx}+u_{yy}=u_{tt}$ amounts to finding all solutions of the system of two equations consisting of the wave equation and the first order equation $(u_x)^2+(u_y)^2=(u_t)^2$. The method referred to in the title is that of N. P. Erugin [Leningrad Gos. Univ. Uč. Zap.. **15**, 1948, pp. 101–134] who considered the wave equation in two and three space variables, and M. M. Smirnov [Leningrad. Gos. Univ. Uč. Zap. **21** (1950), 127–202] who considered the wave equation in four space variables. The simplification referred to in the title consists essentially of making use of the "method of variation of constants", as used by Lagrange in deriving a general integral of the first order partial differential equation $F(x, y, u, u_x, u_y)=0$ from a complete integral.

J. B. Diaz (Cambridge, Mass.).

Galonen, L. M. On functionally invariant solutions of partial differential equations of second order of ultrahyperbolic type. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. **32** (1955), 179–182. (Russian)

[Cf. the preceding review.] The author determines the functionally invariant solutions of the ultrahyperbolic

equation $u_{xx}+u_{yy}=u_{zz}+u_{tt}$ (that is, all solutions of the system consisting of this equation plus the first order equation $(u_x)^2+(u_y)^2=(u_z)^2+(u_t)^2$). *J. B. Diaz.*

Pignedoli, Antonio. Sull'aspetto fisico-matematico del problema delle pile atomiche. Teoria matematica delle vicissitudini subite dai neutroni veloci in mezzi moderatori. Rend. Sem. Mat. Univ. Padova **25** (1956), 250–272.

Diffusion theory for neutrons in atomic piles leads to certain special, but quite intricate, types of boundary and initial-value problems for parabolic differential equations. The author considers monoenergetic neutrons emitted from a distributed source which can vary with time. The neutrons are degenerated, without capture, to low (thermal) energies by collisions with the atoms of the moderator. The differential equations assumed correspond roughly to the physical approximation known as "age theory" [R. E. Marshak, Rev. Mod. Phys. **19** (1947), 185–239], the age parameter and the time being treated as independent variables. The main purpose is to treat the degeneration process for the fast neutrons as a soluble problem, for different geometrical conditions of the moderator. Taking a suitably large value of the age parameter in the resulting density function, the latter would be used as a source function for the thermal neutrons. Only the first part of this program is discussed in detail.

E. L. Hill (Minneapolis, Minn.).

Vaněk, Jiří. Theory of elastic waves produced by a spherical source for generalized boundary conditions. Czechoslovak J. Phys. **6** (1956), 303–309. (Russian summary)

The author considers the equation

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=(\lambda+2\mu)\ \text{grad div}\ \mathbf{u}-\mu\ \text{curl curl}\ \mathbf{u},$$

where $\mathbf{u}$ is the displacement. Two types of boundary conditions on the surface of a sphere of radius a are considered. In one, the displacement is a given function of the spherical angles (ϑ, φ) on the sphere multiplied by an arbitrary function of the time. In the second the stresses are given in a similar form. The method employed for solution of these problems involves the use of the Laplace transform in time and the solution of the resulting problem by expansion in a series in spherical harmonics.

H. Feshbach (Cambridge, Mass.).

Vasilache, Serge. Sur la problème de Cauchy pour l'équation

$$\frac{\partial^2\Delta u}{\partial t^2}+\mu^2\frac{\partial^2 u}{\partial z^2}-\lambda^2 u=f(x, y, z).$$

Rev. Math. Pures Appl. **1** (1956), no. 2, 51–60.

The quantities μ and λ appearing in the differential equation are numerical parameters. The author carries out a Laplace transformation in time and a Fourier transformation in space to arrive at a formal expression for u as a sum of a volume integral in space time and surface integrals on the initial hyperplane $t=0$. The integrands are obtained in closed form only when $\lambda=0$ or $\mu=0$. In the former case, the formula is identical with one found in a different manner by S. L. Sobolev [Izv. Akad. Nauk SSSR. Ser. Mat. **18** (1954), 3–50; MR **16**, 1029].

R. McKelvey (Boulder, Colo.).

See also: Bers, p. 470; Lewy, p. 473; Kuranishi, p. 474; Roelcke, p. 476; Brousse, p. 480; Churchill, p. 480; Bochner, p. 484; De Giorgi, p. 489; Gelfand et Šilov, p. 493; Bielecki, p. 494; Moisil, p. 498; Gloden, p. 504; Wegner, p. 525; Kopzon, p. 526; Šemyakin, p. 527; Yuščenko, p. 527; Stoker, p. 528; Fieber, p. 537; Selig, p. 537.

Difference Equations, Functional Equations

Aczél, J. Über Additions- und Subtraktionstheoreme. Publ. Math. Debrecen **4** (1956), 325–333.

The author considers the functional equations (1) $f(x+y)=F(f(x), f(y))$ and (2) $f(x-y)=G(f(x), f(y))$ for real x and y. After exploring the restrictions imposed on the functions F and G by the requirements that their respective equations possess continuous, non-constant solutions, the author proves the following theorems.

Theorem 1: Eq. (1) has a non-constant continuous solution for all real x and y if and only if there exists an open interval under which the operation $u\circ v=F(u, v)$ constitutes a continuous group. Further, any continuous non-constant solution of (1) is strictly monotone.

Theorem 2: If (1) has a non-constant continuous solution, every solution of (1) which on a set of positive measure has a measurable majorant (or minorant) is continuous and, if non-constant, strictly monotone. For any solution $f(x)$ the functions $f(cx)$ are solutions, and furthermore the only solutions of the prescribed kind.

Theorem 3: If (2) has a solution continuous for all x, the solution, if non-constant, is strictly monotone. Such a non-constant, strictly monotone, continuous solution exists if and only if there exists an open interval on which the operation $u/v=G(u, v)$ is continuous, transitive, involutory, and possesses a right-handed unit: $u/e=u$.

Theorem 4: Identical with Th. 2 except for (2) throughout.

Reference is given to earlier works of the author, Caccioppoli, Ostrowski, Kestelman, and others.

P. E. Guenther (Cleveland, Ohio).

See also: Magnus, p. 523.

Calculus of Variations

Picone, Mauro. Su un elementare problema di estremo. Rev. Un. Mat. Argentina **17** (1955), 173–184 (1956).

The author considers a simple integral problem in space of $(r+1)$ dimensions, with fixed end-values, and with an integrand which is a quadratic form in the derivatives only, the coefficients being continuous functions of x. Sufficient conditions for existence and for non-existence of a minimum are given, and cases are considered in which the lower bound of the integral can be calculated although the minimum does not exist. *L. M. Graves.*

Gautschi, Walter. Bemerkung zu einer notwendigen Bedingung von Picone in der Variationsrechnung. Comment. Math. Helv. **31** (1956), 1–4.

The author gives a proof of this condition which assumes only continuity of the integrand. The condition is too complicated to state here. The variational problem considered may involve derivatives of higher order and minimizing curves having points on the boundary of the domain of the integrand function. [See Picone, Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8) **4** (1954), 137–176; MR **16**, 266.] *L. M. Graves.*

Gautschi, Walter. Una estensione agli integrali doppi di una condizione di Picone, necessaria per un estremo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 283–289.

This note considers extremizing surfaces for a double integral $\iint f(x, y, z, z_x, z_y)dxdy$, which may have elements on the boundary of the domain of the integrand function f, and states a necessary condition analogous to one given by Picone [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8) **4** (1954), 137–176; MR **16**, 266] for simple integrals. The conditions of Legendre and of Weierstrass are derived as corollaries. *L. M. Graves.*

De Giorgi, Ennio. Sull'analiticità delle estremali degli integrali multipli. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 438–441.

The author states a theorem on the analyticity of the extremals of an r-tuple integral

$$\int f(\partial u/\partial x_1, \cdots, \partial u/\partial x_r)dx_1 \cdots dx_r$$

when the integrand $f(p_1, \cdots, p_r)$ has continuous second partial derivatives f_{hk} satisfying

$$\mu_1|\lambda|^2 \leqq \sum f_{hk}\lambda_h\lambda_k \leqq \mu_2|\lambda|^2$$

throughout $(p_1, \cdots, p_r)$-space, where μ_1 and μ_2 are positive constants. *L. M. Graves* (Chicago, Ill.).

Fleming, Wendell H. An example in the problem of least area. Proc. Amer. Math. Soc. **7** (1956), 1063–1074.

It is well-known that given a simple closed rectifiable curve C in R^3 and a topological type τ of surfaces (Euler characteristic and character of orientability) there exists a surface of least area bounded by C and of type τ. The author gives an example of a curve C such that the analogous problem of least area with unrestricted topological types has no solution of finite topological type. The construction starts with a curve Γ_0 such that the problem of least area for surfaces of the type of the disc is greater than the absolute minimum area. C is essentially obtained by joining subarcs of images of Γ_0 under countably many similarity transformations of R^3. A special form of the existence theorem quoted above and an approximation theorem by L. C. Young are used.

K. Krickeberg (Madison, Wis.).

Behrbohm, Hermann. Zur Herleitung der Eulergleichungen des brachystochronen endpunktgebundenen Steigfluges in vertikaler Ebene bei verschiedenen Endbedingungen. Z. Flugwiss. **4** (1956), 373–382.

Seeger, Alfred. A variational principle for conduction phenomena in the presence of a magnetic field or asymmetric scattering mechanisms. Canad. J. Phys. **34** (1956), 1278–1280.

An iterative method is given for solutions of the Boltzmann equation when it includes a magnetic field or other small terms which make the equation non self-adjoint.

D. Falkoff (Waltham, Mass.).

See also: Fleming and Young, p. 503; Santaló, p. 505; Ku and Su, p. 508; Su, p. 508; Hodge, p. 527; Glaser, p. 535; Davies, p. 540; Moses, p. 540; Ingarden, p. 542.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Schöneborn, Heinz. Über den Zusammenhang zwischen Dualitäts- und Vollständigkeitseigenschaften bei gewissen topologischen Abelschen Gruppen. Math. Z. **65** (1956), 429–441.

The author continues a detailed study of the class of groups introduced in two previous papers [Math. Z. **59** (1954), 455–473; **60** (1954), 17–30; MR **15**, 932]. He is concerned with conditions for duality to hold and relations between properties of a group and those of its character group. *I. Kaplansky* (Princeton, N.J.).

Munn, W. D. Matrix representations of semigroups. Proc. Cambridge Philos. Soc. **53** (1957), 5–12.

Complete sets of inequivalent irreducible representations of a finite semigroup S over a field F are given under various conditions. One condition is that the algebra of S over F be semisimple. Other conditions have to be stated at greater length; some of them are given here in abbreviated form.

Let $(x)_{ij}$ designate the $m \times n$ matrix which has x in the (i, j)th position and zeros everywhere else. The element x is from the group G. The group G does not contain the zero, but $0g=0=g0$ defines the product of 0 with an element g from G; the elements of G together with 0 form a semigroup $G(0)$. If P is an $n \times m$ matrix over $G(0)$, then we can define $(x)_{ij} \circ (y)_{kr} = (x)_{ij}P(y)_{kr} = (xp_{jk}y)_{ir}$. Under this definition these matrices form a semigroup $S_{mn}(G, P)$. Rees [same Proc. **36** (1940), 387–400; MR **2**, 127] has shown that every finite simple semigroup is isomorphic with a "regular" matrix semigroup $S_{mn}(G, P)$. By regular is meant that no row or column of P is all zeros. Conversely $S_{mn}(G, P)$ is simple. A complete set of inequivalent irreducible representations for $S_{mn}(G, P)$ is given.

An "inverse" or Vagner semigroup is one in which for each element a there exists a unique x such that $axa=a$ and $xax=x$, and furthermore pairs of idempotents commute. A Vagner semigroup is isomorphic to an $S_{nn}(G, E)$, where E is the identity matrix. A complete set of inequivalent irreducible representations is given for this semigroup over a field of characteristic zero or a prime not dividing the order of G. *H. Campaigne.*

Nomizu, Katsumi. Un théorème sur les groupes d'holonomie. Nagoya Math. J. **10** (1956), 101–103.

It is shown, for a differentiable principal fiber bundle B that if the structural group can be reduced to the connected Lie subgroup H then there exists a connection on B for which the restricted holonomy group is H.

W. Ambrose (Cambridge, Mass.).

Rudin, Walter. The automorphisms and the endomorphisms of the group algebra of the unit circle. Acta Math. **95** (1956), 39–55.

Let C be the circle group (multiplicative group of all complex numbers of absolute value 1), J the additive group of all integers, $L_1(C)$ the group algebra of C, and $M(C)$ the convolution algebra of all complex-valued, countably additive, bounded Borel measures on C. A

subset N of J is called a P-set if N can be made periodic by adding or removing a finite number of elements. Let N be any non-void subset of J and let t be a mapping of N into J. Consider the following property (A) of certain t's. (A): There is a mapping s of J into J and a positive integer q such that (A1) $t(n)=s(n)$ except on a finite subset of N; (A2) for every $n \in J$, $s(n+q)+s(n-q)=2s(n)$; (A3) for every $n \in J$, $s(n+q)\neq s(n)$. Given a mapping t of N into J, and a complex function $a(n)$ defined on J, let $a(t(n))$ be the function on J that is zero if $n \notin N$ and equal to $a(t(n))$ if $n \in N$. The mapping t of N into J is said to carry $L_1(C)$ into $L_1(C)$ if $a(t(n))$ is the Fourier transform of some element of $L_1(C)$ whenever $a(n)$ is the Fourier transform of an element of $L_1(C)$. The mapping t is said to carry $L_1(C)$ or $M(C)$ into $M(C)$ under obvious similar conditions. Theorem I. The mapping t carries $L_1(C)$ into $L_1(C)$ if and only if N is a P-set and t has property (A). This is the (much more difficult) analogue of a theorem of Leĭbenzon [Uspehi Mat. Nauk (N.S.) **9** (1954), no. 3(61), 157–162; MR **16**, 241] and Kahane [C. R. Acad. Sci. Paris **240** (1955), 36–37; MR **16**, 467] concerning functions on C that carry $L_1(J)$ into $L_1(J)$. Theorem II. Let t be a mapping of N into J, where $N \subset J$. Then the following are equivalent: (i) t carries $L_1(C)$ into $M(C)$; (ii) N is a P-set and (A1), (A2) hold; (iii) t carries $M(C)$ into $M(C)$. Theorems I and II enable the author to identify all endomorphisms and automorphisms of $L_1(C)$ and $M(C)$. The endomorphisms of $L_1(C)$ are given by $a(n)\to a(t(n))$ ($a(n)$ is the Fourier transform of a generic element of $L_1(C)$), where the domain N of t is a P-set and t satisfies (A). The automorphisms of $L_1(C)$ and of $M(C)$ are just the endomorphisms for which $N=J$ and t is one-to-one. As a corollary, the author shows that absolutely continuous measures, continuous measures, and measures with no singular components go into measures of the same sorts under any automorphisms of $M(C)$. *E. Hewitt.*

Ionescu Tulcea, Cassius. Fonctions de type positif. C. R. Acad. Sci. Paris **243** (1956), 1389–1392.

Let G be a locally compact group and S a locally compact subspace of G. Denote by μ the restriction to S of the left invariant Haar measure on G. The subspace S is assumed to satisfy the following conditions: (1) S is a semi-group in G and contains the identity element e of G. (2) There is given a continuous mapping $x\to x^+$ of S into S such that $(xy)^+=y^+x^+$ and $x^{++}=x$ for all $x, y, z \in S$. (3) If the set A is a locally negligible subset of S, then so is A^+. (4) If V is a compact neighborhood of e in S, then $\mu(V)>0$.

A complex-valued function f which is μ-measurable and μ-bounded on S is said to be of "positive type" if

$$\int\int_S f(x^+y)\,d\bar{\nu}(x)d\nu(y)\geqq 0$$

for every Radon measure ν on S with compact support and base μ (i.e. $\nu=g\cdot\mu$, where g is locally integrable). In this situation the author states a number of theorems which are extensions of classical results for functions of positive type on a locally compact group.

C. E. Rickart (New Haven, Conn.).

Wallace, Alexander Doniphan. Ideals in compact connected semigroups. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 535–539.

Throughout this review, S stands for a compact topological semigroup, E the set of idempotents of S, and K the minimal ideal (which always exists for S compact). For $e \in E$, there is a group H_e which is maximal with respect to e being its identity. Let $J(a)=a\cup Sa\cup aS\cup SaS$, $J_a=\{x: J(a)=J(x)\}$, and $I(a)=J(a)\setminus J_a$. Thus $J(a)$ and I_a are ideals. Several results are proved showing the position of these sets relative to one another and other subsets of S. The main theorem is as follows: Let S be a continuum such that no retract of S cuts S. If $a \in E\setminus K$, and if some closed subset of H_e cuts S, then (i) H_e cuts S into two components, $I(e)$ and $S\setminus J(e)$; (ii) $I(e)^*=J(e)=eS=Se$ and e is in the center of S; (iii) the boundary of $I(e)$ is $H_e=J_e$.

P. S. Mostert (New Orleans, La.).

See also: Ono, p. 469; Bochner, p. 484; Silverman, p. 492; Allamigeon, p. 496; Lashof, p. 497; Kobayashi, p. 503; Egorov, p. 506; Hano and Ozeki, p. 507; Lichnerowicz, p. 508; Hirzebruch, p. 509; Rosenlicht, p. 514; Barsotti, p. 514; Karle and Hauptman, p. 524.

Lie Groups and Algebras

Harish-Chandra. Representations of semisimple Lie groups. V. Amer. J. Math. **78** (1956), 1–41.

Let G be a connected semisimple Lie group and A a Cartan subgroup of G. Under the assumption that the image of G in the adjoint group is compact the author has studied in detail certain irreducible representations of the Lie algebra of G and shown that they correspond to representations of the group G [same J. **77** (1955), 743–777; MR **17**, 282]. In the present paper these irreducible representations of G are obtained directly by letting G act on certain Hilbert spaces consisting of holomorphic functions on a suitable complex analytic manifold. For the case $G=\mathrm{SL}(2, R)$ these representations constitute the discrete series of irreducible unitary representations discovered by V. Bargmann [Ann. of Math. (2) **48** (1947), 568–640; MR **9**, 133] who described them explicitly by letting $\mathrm{SL}(2, R)$ act on certain Hilbert spaces of functions holomorphic in the unit disc. *F. I. Mautner.*

Harish-Chandra. Representations of semisimple Lei groups. VI. Integrable and square-integrable representations. Amer. J. Math. **78** (1956), 564–628.

Let G be a connected semisimple Lie group and Z its center. *If* π is an irreducible unitary representation of G with Hilbert space H then $\pi(z)=\gamma_\pi(z)\pi(1)$ for $z \in Z$, where γ_π is a unitary character of Z. Let $x\to x^*$ denote the natural map of G onto $G/Z=G^*$. If $\varphi,\psi \in H$ then $|(\varphi, \pi(x)\psi)|$ is clearly a function on G^*. The author calls π integrable (square integrable) if there exists an element $\psi_0\neq 0$ of H such that $|(\psi_0, \pi(x)\psi_0)|$ is an integrable (square integrable) function on G^* with respect to the Haar-measure on G^*. In this paper examples of such representations are studied. By the Schur-orthogonality relations there exists a positive constant d_π such that $\int_{G^*} |(\varphi, \pi(x)\psi)|^2 dx^* = d_\pi^{-1}|\varphi|^2|\psi|^2$ for all $\varphi, \psi \in H$. This constant d_π has some formal analogy with the degree of an irreducible representation of a compact group. One of the main results of this paper is (under suitable conditions) an expression for d_π which is analogous to Weyl's formula for the degree of an irreducible representation of a compact semisimple Lie group in terms of its highest weight. The proof depends on a detailed comparison at each step between the compact and non-compact cases. For this the author has to go through a study of the algebraic root-structure of certain semisimple Lie algebras. An incidental outcome of this study is a new proof of a

theorem of E. Cartan on the boundedness of certain complex domains which does not depend on the classification of the simple Lie groups.

The results of this paper have been announced in Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 314–317 [MR **17**, 60].

F. I. Mautner (Baltimore, Md.).

Curtis, Charles W. On Lie algebras of algebraic linear transformations. Pacific J. Math. **6** (1956), 453–466.

Let L be a locally finite Lie algebra on a vector space M over a field of characteristic 0. If the associative algebra generated by L has no nil ideals then L is the direct sum of its center and an ideal L_1 containing $[L, L]$ but containing no solvable ideals. In case M is finite dimensional, it is known that $L_1=[L, L]$; this equality need not hold for infinite dimensional M.

Thus if L is locally finite and completely reducible on M, the decomposition above holds. If L is solvable besides, then L is abelian. (However, the hypothesis that a solvable algebra L of linear transformations is locally finite is equivalent to assuming that every element of L is algebraic.)

The principal tool used is the decomposition of every algebraic linear transformation as a sum of a semisimple transformation and a nilpotent one. If each element of L can be so decomposed into transformations belonging to L, and if we assume besides that L is solvable and of denumerable dimension, then L is the (vector-space) direct sum of an abelian subalgebra consisting of semisimple transformations and an ideal (containing $[L, L]$) consisting of all the nilpotent transformations in L. The denumerability hypothesis cannot be dropped.

D. Zelinsky (Princeton, N.J.).

Ree, Rimhak. On generalized Witt algebras. Trans. Amer. Math. Soc. **83** (1956), 510–546.

A generalized Witt algebra is a Lie algebra obtained from a vector space V possessing a finite total additive group G of functionals by forming a vector space L over the base field Φ with basis the set of pairs (e_i, γ), where $\{e_i\}$ is a basis for V and γ runs through G. One defines $[(e_i, \alpha), (e_j, \beta)]=\beta(e_i)(e_j, \alpha+\beta)-\alpha(e_j)(e_i, \alpha+\beta)$. L is in general a simple Lie algebra and generalizes those of Zassenhaus [Abh. Math. Sem. Hansischen Univ. **13** (1939), 1–100] and of Jacobson [Duke Math. J. **10** (1943), 107–121; MR **4**, 187]. The author notes that the generalized Witt algebras are a special case of a class of Lie algebras L having the following property: $L=L(A; D_1, \cdots, D_m)$ is a subalgebra of the derivation Lie algebra of a commutative associative algebra A over Φ, and L contains derivations $D_1, \cdots, D_m$ such that every element of L is uniquely expressible in the form $f_1D_1+\cdots+f_mD_m$, where $f_i \in A$ and fD denotes the derivation $a \to f \cdot D(a)$ of A. In particular, if A is a field and $p>2$ or $m \neq 1$, $L(A; D_1, \cdots, D_m)$ is a simple Lie algebra. In the case where Φ is algebraically closed and $L(A; D_1, \cdots, D_m)$ is simple, $L(A; D_1, \cdots, D_m)$ is a generalized Witt algebra. Further conditions for simplicity over arbitrary fields are obtained and are applied to display new simple Lie algebras over the prime field $GF(p)$. Relations between the automorphism group of $L(A; D_1, \cdots, D_m)$ and that of A are also established. If $L(A; D_1, \cdots, D_m)$ is normal simple, the natural number m is an invariant. *G. Seligman* (New Haven, Conn.).

See also: Nomizu, p. 489; Bernard, p. 498; Grincevičyus, p. 505; Hirzebruch, p. 509; Barsotti, p. 513.

Topological Vector Spaces

Vasilach, Serge. Sur le produit de composition des fonctions et distributions à support dans R^n_+, $n>1$. C. R. Acad. Sci. Paris **243** (1956), 1591–1593.

Dans l'algèbre (pour le produit de composition) des fonctions définies dans R^n, à support dans $(x_1 \geqq 0, \cdots, x_n \geqq 0)$, il n'y a pas de diviseur de zéro. Pour $n=1$, ce résultat est du à Titchmarsh, Crum, Dufresnoy; pour le cas général, au rapporteur et à Mikusiński et Ryll-Nardzewski [Studia Math. **13** (1953), 62–68; MR **15**, 408]. L'A. reprend ici des méthods de Mikusiński et Ryll-Nardzewski pour obtenir une nouvelle démonstration du théorème. *J. L. Lions* (Lawrence, Kan.).

Luxemburg, W. A. J. On a characteristic property of σ-finite measures. Nieuw Arch. Wisk. (3) **4** (1956), 124–126.

A measure space is σ-finite if and only if every length function on the space is reflexive. *I. Halperin.*

Pták, Vlastimil. Two remarks on weak compactness. Czechoslovak Math. J. **5(80)** (1955), 532–545. (Russian summary)

Let X be a complete, locally convex, topological linear space over the real number field. Let $L(X)$ be the space of all continuous linear functionals on X, and let $F(L(X))$ be the space of all linear functionals (not necessarily continuous) on $L(X)$. Let X be embedded in $F(L(X))$ in the usual way. Main theorem. Let B be a bounded closed convex subset of X. Then the following properties of B are equivalent. (1) If $r \in F(L(X))$, r is in the closure of B, and $\{y_n\}_{n=1}^{\infty}$ is any sequence of elements of $L(X)$, then there is a $b \in B$ such that $by_n=ry_n$ $(n=1, 2, 3, \cdots)$. (2) If $\{H_n\}_{n=1}^{\infty}$ is a sequence of closed hyperplanes in X such that $B \cap H_1 \cap \cdots \cap H_n \neq 0$ for $n=1, 2, 3, \cdots$, then $B \cap \bigcap_{n=1}^{\infty} H_n \neq 0$. (3) B is weakly compact. (4) B is weakly pseudo-compact (every continuous real function is bounded). (5) Let $\{y_n\}_{n=1}^{\infty}$ be a sequence of elements of $L(X)$ and $\{\beta_n\}_{n=1}^{\infty}$ a sequence of real numbers. Suppose that for every natural number m and all real numbers $\lambda_1, \cdots, \lambda_m$, the inequality

$$\sum_{j=1}^{m} \lambda_j\beta_j \leqq \sup B\Big(\sum_{j=1}^{m} \lambda_j y_j\Big)$$

holds. Then there is a point $b \in B$ such that $by_n=\beta_n$ for $n=1, 2, 3, \cdots$. This theorem generalizes a result due to Floyd and Klee [Proc. Amer. Math. Soc. **5** (1954), 655–661; MR **16**, 59], as well as earlier results of the author [Studia Math. **14** (1954), 276–284; MR **16**, 595]. The author also proves the following theorem. Let B be a bounded subset of X, and suppose that B has property (1). Then every r in the closure of B in $F(L(X))$ is almost continuous. That is, for every neighborhood U of 0 in X, the restriction of r to the set

$$U^*=\{y: y \in L(X), y(U) \subset [-1, 1]\}$$

is continuous in the weak topology of U^* defined by elements of X. In connection with this notion, see the earlier paper of the author [Czechoslovak Math. J. **3(78)** (1953), 301–364; MR **16**, 262]. *E. Hewitt.*

Amemiya, Ichiro. A generalization of the theorem of Orlicz and Birnbaum. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1956), 60–64.

Let $\Phi(u)$ be a convex and increasing function for $u \geqq 0$ with $\Phi(0)=0$. If $x(t)$, $0 \leqq t \leqq 1$, is a Lebesgue measurable

function we put $m(x)=\int_0^1 \Phi(|x(t)|)dt$. Z. W. Birnbaum and W. Orlicz [Studia Math. 3 (1931), 1–67] proved that $m(x)$ is finite (i.e. $m(x)<\infty$ implies $m(2x)<\infty$) if and only if there exists constants $M>0$, $u_0\geq 0$ such that

$$\Phi(2u)\leq M\Phi(u)$$

for $u\geq u_0$. The author extends this result to arbitrary monotone complete modulars $m(x)$ defined on non-discrete semi-ordered linear spaces R [see M. Nakano, Modulared semi-ordered linear spaces, Maruzen, Tokyo, 1950, MR 12, 420]. He calls a modular $m(x)$ semi-upper bounded if there exist constants ε, $\gamma>0$ such that $m(2x)\leq\gamma m(x)$ for all $m(x)\geq\varepsilon$. The main result is that a finite modular is semi-upper bounded. {Reviewer's remark. The converse of this result is in general not true.} The paper closes with a discussion of finite modulars of special form on discrete spaces R. *W. A. J. Luxemburg.*

Kyner, Walter T. A generalization of the Borsuk and Borsuk-Ulam theorems. Proc. Amer. Math. Soc. **7** (1956), 1117–1119.

Let E be a real linear topological vector space with a bounded convex neighborhood of the zero element 0. E is supposed to be not finite dimensional. Let V be a convex subset of E which is symmetric with respect to 0, and S its boundary. A continuous map $f: S\to E$ is called a compact displacement if $f=I-F$, where I is the identity map and F a compact map, i.e. maps S onto a set whose closure is compact. The main results of the paper are contained in the following two theorems concerning antipodal maps $f(x)$, i.e. maps with $f(-x)=-f(x)$. Theorem I. Let f be a compact antipodal displacement of S. Then the topological order of 0 with respect to the image of S is odd. Theorem II. A compact antipodal map F of S, whose range is closed, maps at least one pair of antipodal points into 0. Theorem I generalizes to infinite dimensional spaces a theorem by Borsuk, and Theorem II generalizes a consequence of a theorem by Borsuk-Ulam [see Alexandroff and Hopf, Topologie, Bd. 1, Springer, 1935, pp. 483, 486]. Finally, using Theorem II, it is shown that a compact linear map of a reflexive Banach space into itself has a non-trivial nullspace if and only if the image of the unit sphere is closed. *E. H. Rothe.*

Ball, B. J. The normality of the product of two ordered spaces. Duke Math. J. **24** (1957), 15–18.

If X is a linearly ordered space (with the interval topology), let X^+ denote the compact space obtained by adding end points (if necessary) to its Dedekind completion. It is shown that $X\times X^+$ is normal if and only if X is paracompact. Let X, Y denote respectively a compact and a locally compact linearly ordered space. A necessary and sufficient condition (depending on X and Y) that $X\times Y$ be normal is given. The proofs depend heavily on a characterization of linearly ordered paracompact spaces due to L. Gillman and the reviewer [Trans. Amer. Math. Soc. 77 (1954), 340–362; MR 16, 156]. *M. Henriksen.*

Silverman, Robert J. Invariant linear functions. Trans. Amer. Math. Soc. **81** (1956), 411–424.

Let V be a partially ordered linear space over the real number field in which every bounded set has a least upper bound. Let Y be a linear space and X a linear subspace of Y. Let G be a semigroup of linear operators on Y, and p a positive-homogeneous subadditive mapping of Y into V such that $p(gy)\leq p(y)$ for all $y\in Y$ and $g\in G$. Let f be a linear mapping of X into V such that $f(gx)=f(x)$ and $f(x)\leq p(x)$ for all $g\in G$ and $x\in X$. The collection $[Y, X, V, p, f, G]$ is said to have the Hahn-Banach extension property (HBEP) if there is a linear extension F of f over Y such that $F(y)\leq p(y)$ and $F(gy)=F(y)$ for all $y\in Y$ and $g\in G$. An abstract semigroup $\mathfrak{G}$ (set with a binary associative operation) is said to have the HBEP if every $[Y, X, V, p, f, G]$ has the HBEP, where G is an arbitrary representation or anti-representation of $\mathfrak{G}$ by linear operators on Y.

The author first shows that the HBEP for $\mathfrak{G}$ is equivalent to another property, called the monotone extension property (MEP). (This property asserts essentially that invariant monotone mappings can be extended from subspaces to superspaces.) The author next shows that a large class of semigroups have the HBEP: commutative semigroups; finite groups; semigroups having certain semigroups with HBEP; and so on.

The last section deals with invariant means on the space $m(\mathfrak{G})$ of bounded real-valued functions on $\mathfrak{G}$. The author shows that if $\mathfrak{G}$ has the HBEP, then there is an invariant mean on $m(\mathfrak{G})$. *E. Hewitt.*

Nef, Walter. Über lineare Formen, die gegenüber einer abelschen Gruppe linearer Transformationen invariant sind. Arch. Math. **7** (1956), 250–258.

Let $\mathfrak{M}$ be a partially ordered real vector space and let Γ be a group of order-preserving linear transformations on $\mathfrak{M}$. Question 1: If x_0 is in $\mathfrak{M}$, does there exist an invariant positive linear functional F on $\mathfrak{M}$ such that $F(x_0)=1$? Question 2: If F_0 is an invariant positive linear functional on an invariant subspace of $\mathfrak{M}$, does F_0 have an invariant positive linear extension to $\mathfrak{M}$? (Invariance refers, of course, to the group Γ.) The main purpose of the paper is to answer these two questions in case the group Γ is abelian. The answer to Question 1 is yes if and only if there exists a positive linear functional f on $\mathfrak{M}$ such that (a) $f(x^\sigma)$ is bounded, as a function of the element σ of Γ, for each x in $\mathfrak{M}$, and (b) $f(x_0^\sigma)\geq 1$ for each σ in Γ. The answer to Question 2 is yes if and only if F_0 has a positive extension f to $\mathfrak{M}$ with the property (a). *P. R. Halmos.*

Raševskiĭ, P. K. Multidimensional δ-functions and differential-geometric objects. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4(66), 145–152. (Russian)

A one-dimensional δ-function is a L. Schwartz distribution which in the language of physicists is a function of a real variable that is zero everywhere except at one point X_0 and whose integral equals 1. The present paper generalizes this idea to n-space in which an m-dimensional surface plays the part of the point X_0 or, as the author says "similarly to the situation in which the simplest δ-functions of one argument are "concentrated" in one point, our many-dimensional δ-functions are "concentrated" on a m-dimensional surface $U_m\subset X_n$.

The author considers an n-dimensional manifold X_n, local coordinates $x^1, \cdots, x^n$, which he assumes to have a scalar density, $\gamma(x^1, \cdots, x^n)$ in the local coordinates, and therefore a volume element $\gamma dx^1\cdots dx^n$. A relative covariant hyper-vector of class v is given in local coordinates by functions

$$\Phi_*, \Phi_{i_1}, \Phi_{i_1 i_2}, \cdots, \Phi_{i_1 i_2\ldots i_v}, \cdots, \Phi_{i_1 i_2\ldots i_v}$$

where these transform like $\partial^s\Phi_*/\partial x^{i_1}\cdots\partial x^{i_s}$, $s=0, 1, \cdots, v$, with Φ_* a scalar density. A relative contravariant hyper-vector A^*, A^{i_1}, $A^{i_1 i_2}$, $\cdots$, $A^{i_1\ldots i_v}$, $\cdots$, $A^{i_1 i_2\ldots i_s}$ s ialso introduced, the transformation under change of coordinates being contragradient to the covariant case.

These symbols are assumed to be symmetric in the indices, and abbreviating the collection of indices by I we may write

$$A^I\Phi_I=A^*\Phi_*+A^{i_1}\Phi_{i_1}+A^{i_1i_2}\Phi_{i_1i_2}+\cdots+A^{i_1\cdots i_n}\Phi_{i_1\cdots i_n}.$$

This is an invariant or scalar function defined on X_n. Let U_m denote a regularly imbedded m-dimensional submanifold of X_n, given locally by $x_i=x_i(u^1, \cdots, u^m)$, $i=1, \cdots, n$, and with regular boundary Γ_{m-1}. Let a relative contravariant hyper-vector A^I be given on U_m (under change of local parameters $u^1, \cdots, u^m$ it will be multiplied by the Jacobian); then in the terminology of the author this determines a multidimensional δ-function on X_n. This δ-function and the scalar density γ determine a linear functional on sufficiently differentiable functions φ defined in a neighborhood of U_m, namely

$$A(\varphi)=\int_{U_m} A^I(\gamma\varphi)_I du^1\cdots du^m,$$

where $(\gamma\varphi)_I$ are the partial derivatives of the scalar density $\gamma\varphi$. A^I will be called reducible if $A(\varphi)=0$ for every φ which vanishes in some neighborhood of Γ_{m-1}. The author devotes most of the note to showing that a necessary and sufficient condition for reducibility of the δ-function A^I is that it be in some sense the divergence of an α field of hyper-vectors $B^{\alpha K}$, which is a relative contravariant hyper-vector of order $v-1$ in the indices K, as defined above, but relative to the index α transforms by

$$B^{\alpha' K}\to\left|\frac{\partial(u^1, \cdots, u^m)}{\partial(u^{1'}, \cdots, u^{m'})}\right|\cdot\frac{\partial u^{\alpha'}}{\partial u^\alpha}B^{\alpha K}.$$

More precisely A^I is reducible if and only if there exists a field $B^{\alpha K}$ such that

$$A^I(\gamma\varphi)_I=\frac{\partial}{\partial u^\alpha}(B^{\alpha K}(\gamma\varphi)_K).$$

The author concludes with a definition of the notion of derivative of the δ-function A^I. *W. M. Boothby.*

Gelfand, I. M.; et Šilov, G. E. Quelques applications de la théorie des fonctions généralisées. J. Math. Pures Appl. (9) **35** (1956), 383–413.

The authors apply the ideas of L. Schwartz's theory of distributions by replacing Schwartz's spaces $\mathfrak{D}$ and $\mathfrak{S}$ by certain subspaces of $\mathfrak{S}$ which, when they are suitably topologized, turn out to be either Fréchet-Montel spaces, or strict inductive limits of such spaces. For instance, given a double sequence (m_{kp}) of positive numbers, they consider the subspace $S_{m_{kp}}$ of $\mathfrak{S}$ consisting of functions φ satisfying the system of inequalities $|x^k\varphi^{(p)}(x)|\leq CA^kB^pm_{kp}$ (where A, B, C depend on φ); if $m_{kp}=k^{k\alpha}p^{p\beta}$, the space is written S_α^β and can be proved to be non trivial if $\alpha+\beta\geq 1$, the pairs $(0, 1)$ and $(1, 0)$ being excepted. They also consider the space Z_μ^ν consisting of entire functions $\varphi(x+iy)$ verifying inequalities of the form

$$|\varphi(x+iy)|\leq C\exp(-A|x|^\mu+B|y|^\nu),$$

where A, B, C depend on φ ($\mu>1$, $\nu>1$); the spaces $S_\alpha^{1-\alpha}$ and Z_p^p with $p=1/\alpha$ are isomorphic. The images of these spaces by the Fourier transform are determined; for instance the image of S_α^β is S_β^α, the image of Z_μ^ν is $Z_{\nu'}^{\mu'}$ $\left(\text{with } \frac{1}{\lambda}+\frac{1}{\lambda'}=1\right)$. The duals of these spaces provide "generalized functions" which in general are not distributions, and which can be handled by the same methods; in particular they are indefinitely differentiable, and it is even possible to apply to them differential operators of infinite order which are not always meaningful when applied to distributions. Various applications of these ideas are mentioned by the authors (most of which are being treated in detail in other publications); Cauchy problems for partial differential or difference-differential equations, theorems of Phragmén-Lindelöf type for solutions of such equations. Finally, when a scalar product is defined in one of the spaces ϕ they consider, so that ϕ may be imbedded into a Hilbert space H, then ϕ is naturally imbedded in its dual ϕ', and the authors show that under certain restrictions, an operator A on ϕ which can be extended to a self-adjoint operator in H, has a complete system of "eigenvectors" in ϕ'. *J. Dieudonné.*

Ravetz, J. R. Distributions defined as limits. I. Distributions as derivatives; continuity. Proc. Cambridge Philos. Soc. **53** (1957), 76–92.

Soit g_n une suite de fonctions continues sur R. Soit $\mathscr{D}$ l'espace des fonctions indéfiniment différentiables à support compact; la suite g_n est dite régulière si, pour tout $\varphi\in\mathscr{D}$, la suite

$$g_n\varphi=\int g_n(x)\varphi(x)\,dx$$

converge; soit $G\varphi$ la limite; G est dite distribution sur R. (On retrouve bien entendu les distributions de L. Schwartz.) Partant de cette définition, l'A. donne une démonstration élémentaire du résultat: toute distribution est localement une dérivée de fonction continue.

J. L. Lions (Lawrence, Kan.).

Chakravarty, N. K. On symbolic calculus of two variables. Bull. Calcutta Math. Soc. **47** (1955), 239–247.

The author gives three rules for obtaining the two-dimensional operational images of functions which depend, essentially, only on the product or quotient of the variables. The rules are obtained by formal computation and are accompanied by examples. *A. Erdélyi.*

Wright, Fred B. Semigroups and submodular functions. Michigan Math. J. **3** (1955–1956), 169–172.

The structure of open additive semigroups of complex numbers was investigated by Hille and Zorn [Ann. of Math. (2) **44** (1943), 554–561; MR **5**, 40] who obtained a characterization of them in terms of subadditive functions. Rosenbaum [Duke Math. J. **17** (1950), 227–247; MR **12**, 164] extended this to Euclidean n-space. The author [Proc. Amer. Math. Soc. **7** (1956), 309–311; MR **17**, 1107] gave an extension to locally compact abelian groups. The present paper studies an analogous problem for arbitrary real linear topological spaces. Let E be such a space, and f a function from E to the real numbers with $+\infty$ and $-\infty$ adjoined. f is called submodular if $f(x+y)\leq\max(f(x), f(y))$, provided either both $f(x)$, $f(y)$ are finite, or both are $+\infty$, or both are $-\infty$. The main result is that the submodular functions are in one-to-one correspondence with the radiant semigroups of E. A semigroup S is called radiant if it is open and if c_1x, $c_2x\in S$, where $c_1<c_2$ are real numbers, implies $cx\in S$ for all $c_1<c<c_2$. *A. Shields* (Ann Arbor, Mich.).

Inaba, Mituo. Differential equations in coordinated spaces. Kumamoto J. Sci. Ser. A. **2** (1955), 233–243.

Soit (E_n) une suite d'espaces localement convexes, E un sous-espace de l'espace produit $\prod E_n$; on suppose E muni d'une topologie localement convexe ayant les propriétés suivantes: pour tout $x\in E$, et tout n, soit $x^{(n)}$ le

point dont les n premières coordonnées sont celles de x, les autres 0; on suppose alors qu'il y a un système fondamental $\mathfrak{U}$ de voisinages convexes de 0 dans E tel que: pour tout $U \in \mathfrak{U}$, la relation $x \in U$ implique $x - x^{(n)} \in U$ dès que n est assez grand; pour tout $x \in E$, la suite $(x^{(n)})$ converge vers x. Soit alors $x' = f(t, x)$ une équation différentielle où $x \in E$, $t \in I$ (intervalle de la droite réelle), $f(t, x) \in E$; l'auteur montre que si f satisfait à une condition de Lipschitz (i.e. s'il existe une fonction intégrable $\mu(t)$, telle que pour tout voisinage $U \in \mathfrak{U}$, la relation $x_1 - x_2 \in U$ implique $f(t, x_1) - f(t, x_2) \in \mu(t)U$) et si en outre l'équation différentielle admet une solution vérifiant une condition initiale donnée, alors cette solution est unique. Il déduit ce théorème d'un résultat plus général sur la dépendance continue des solutions (supposer exister) relativement au second membre. Il montre aussi comment, en supposant que f satisfasse à une condition de Lipschitz et en outre qu'il y ait un théorème d'existence pour les équations différentielles à valeurs dans chaque espace E_n, on peut approcher uniformément la solution de $x' = f(t, x)$ par les solutions de $x' = f^{(n)}(t, x)$, ou de $x' = f(t, x^{(n)})$, ou de $x' = f^{(n)}(t, x^{(n)})$. *J. Dieudonné.*

Bielecki, A. Une remarque sur la méthode de Banach-Cacciopoli-Tikhonov dans la théorie des équations différentielles ordinaires. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 261–264.

Bielecki, A. Une remarque sur l'application de la méthode de Banach-Cacciopoli-Tikhonov dans la théorie de l'équation $s = f(x, y, z, p, q)$. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 265–268.

Con due esempi, l'uno relativo all'equazione ordinaria $y' = f(x, y)$, l'altro relativo all'equazione a derivate parziali $s = f(x, y, z, p, q)$ si mostra come nell'applicare il teorema di Banach-Caccioppoli, sugli elementi uniti delle trasformazioni funzionali, allo studio dei teoremi di esistenza ed unicità, una opportuna scelta della metrica in uno spazio funzionale ausiliario consenta di eliminare le restrizioni, sul campo di esistenza delle soluzioni, che si presentano quando il metodo si applica con l'ordinaria metrica lagrangiana. *C. Miranda* (Napoli).

See also: Edmondson, p. 461; Hewitt and Zuckerman, p. 465; Evgrafov, p. 472; Narain, p. 481; Višik, p. 487; Ree, p. 491; Misonou, p. 494; Rosen, p. 495.

Banach Spaces, Banach Algebras

Helgason, S. Multipliers of Banach algebras. Ann. of Math. (2) **64** (1956), 240–254.

This paper treats representation theory for commutative Banach algebras with special reference to harmonic analysis. The approach is based on the introduction by the author of two new algebras, the derived algebra A_0 and the multiplier algebra A_1, associated with a given algebra A, which are closely related to A but more tractable. They are determined explicitly for certain convolution algebras on locally compact abelian groups.

The derived algebra A_0 may be defined as the subalgebra of A consisting of elements x in A such that the operation on A of multiplication by x is continuous from A in the spectral radius norm to A in its given norm, i.e. $\|x\|_0 = \sup_y \|xy\| / \lim_n \|(yy^*)^n\|^{(1/2n)}$ is finite. For a semi-simple algebra with an adjunction operation and an approximate identity, A_0 is again of the same type as a Banach algebra relative to the norm $\|x\|_0$, has the same spectral norm relative to both $\|\cdot\|_0$ and the original norm, and has maximal ideal space M_0 essentially an open subset of the maximal ideal space M on A, both in the weak *- and Stone topologies. It is simpler than A notably in that the corresponding class of continuous functions vanishing at infinity on M_0, $A_0(M_0)$, is an ideal in the algebra of all such functions. A typical example is: if $A = L_1(G)$ under convolution as multiplication, where G is locally compact abelian, $A_0 = L_2(G)$ when G is compact, while $A_0 = \{0\}$ when G is non-compact and connected.

A multiplier for A is defined as a complex-valued function on M, multiplication by which leaves invariant $A(M)$. When A is weakly complete (sequentially), all bounded continuous functions on M_0 are multipliers for A_0. This result is established by giving a concrete representation for $A_0(M_0)$ as the algebra of all continuous functions on M_0 that are integrable with respect to certain measures on M_0. Specialization of these results to algebras such as $L_p(0, 2\pi)$ yields several classical theorems about Fourier series. *I. E. Segal.*

Misonou, Yosinao. On divisors of factors. Tôhoku Math. J. (2) **8** (1956), 63–69.

The author studies the algebraic structure of factors, making use of his concept of direct product [same J. (2) **6** (1954), 189–204; MR **16**, 1125]. A factor N is called a divisor of a factor M if M is the direct product of N and another factor. After preliminary considerations the author proves the following. All divisors of a normal factor are normal. If M is a finite factor and some divisor of M has property Γ, then M has property Γ. The restricted infinite direct product of finite factors has property Γ and its fundamental group contains all positive numbers. (The definitions of normality, property Γ, and fundamental group are due to Murray and von Neumann [Ann. of Math. (2) **37** (1936), 116–229; **44** (1943) 716–808; MR **5**, 101]. The definition of restricted infinite direct product is due to Takeda [Tôhoku Math. J. **7** (1955), 67–86; MR **17**, 648].) *J. A. Schatz* (Storrs, Conn.).

Del Pasqua, Dario. Sulle coppie di varietà lineari supplementari di uno spazio di Banach. Rend. Mat. e Appl. (5) **15** (1956), 129–139.

The author resumes his exposition of various phenomena concerned with disjointness of linear manifolds in Banach spaces [same Rend. (5) **13** (1955), 406–422; MR **17**, 986]. Various conditions are given that a Banach space should be the topological direct sum of two subspaces. *E. R. Lorch* (New York, N.Y.).

Taylor, Angus E. Extensions of a theorem of Hellinger and Toeplitz. Math. Z. **66** (1956), 53–57.

The theorem of Hellinger and Toeplitz [Math. Ann. **69** (1910), 289–330, pp. 321–322] to which the title refers states that an infinite matrix which maps l^2 into l^2 always represents a bounded linear operator over l^2. By using the closed graph theorem, this theorem and several generalizations are proved. The principal result is the following: Let X be a Banach space, Z a linear Hausdorff space, S a set, and Y a Banach space of functions over S with values in Z. Then if for each $s \in S$, $a(s)$ is a continuous linear function from X to Z with $a(s)\,x \in Y$ for each $x \in X$ and if A is the function from X to Y defined as $a(\cdot)x = y$, A is a bounded linear operator from X to Y. By considering special cases of S, X, Y, Z a general representation theorem is proved and it is shown that in certain cases the existence of the adjoint of an operator implies the continuity of the operator. *R. E. Fullerton.*

Müller, P. H. Zu einer Spektralbetrachtung von Atkinson und Sz.-Nagy. Acta Sci. Math. Szeged **17** (1956), 195–197.

Another proof of a theorem of Atkinson [Acta Math. Acad. Sci. Hungar. 3 (1952), 53–60; MR **14**, 478] and Sz.-Nagy [ibid. 3 (1952), 61–66; MR **14**, 478] that if $V_1, \cdots, V_n$ are completely continuous transformations on a Banach space R into itself, then the characteristic values of $T(\lambda)=I+\sum_{r=1}^{n}\lambda^r V_r$ have no finite limiting point. On the product space $\mathfrak{R}=R\times R\times\cdots\times R$ let $\mathfrak{V}$ be the matrix of transformations on R having the identity above the main diagonal, $-V_n, -V_{n-1}, \cdots, -V_1$ in the last row and zero elsewhere. Then the matrix $\mathfrak{V}^n$ consists of completely continuous transformations and has this property on $\mathfrak{R}$. Hence $\mathfrak{V}$ has similar properties. The characteristic numbers of $\mathfrak{I}-\kappa\mathfrak{V}$ are the same as those of $T(\lambda)$.

T. Hildebrandt (Ann Arbor, Mich.).

Rosen, William G. On invariant means over compact semigroups. Proc. Amer. Math. Soc. **7** (1956), 1076–1082.

Let Σ denote a compact topological semigroup. It is known that Σ contains a unique minimal (two-sided) ideal K, which is closed [Numakura, Math. J. Okayama Univ. **1** (1952), 99–108; MR **14**, 18]. Let $C(\Sigma)$ denote the Banach space of all continuous real-valued functions on Σ. A mean on Σ is a positive continuous linear functional on $C(\Sigma)$ of norm 1. The concept of a right invariant, left invariant, and invariant (under translation) mean is defined in the natural way, as are the analogous concepts of invariant (Borel) measures. The main results of the paper are given below.

A right invariant mean on Σ exists if and only if Σ has a unique minimal left ideal (which then coincides with K). An invariant mean exists on Σ if and only if K is a group. If a right invariant measure m on Σ exists, then K is the unique minimal left ideal of Σ and $m(\Sigma-K)=0$, but the existence of a right invariant measure is not implied by the existence of a unique minimal left ideal. It is shown that if a right invariant mean exist on Σ, then K is the direct product, algebraically, topologically, and in the measure-theoretic sense, of the subsemigroup of idempotents in K (under the multiplication $ab=a$) and a subgroup of K. Finally, it is shown that if Σ has a unique right invariant mean M, then M is (two-sided) invariant.

M. Henriksen (Princeton, N.J.).

See also: Hewitt and Zuckerman, p. 465; Luxemburg, p. 491.

Hilbert Space

Laugwitz, Detlef; and Lorch, Edgar R. Riemann metrics associated with convex bodies and normed spaces. Amer. J. Math. **78** (1956), 889–894.

Let C be a bounded convex body in Hilbert space $\mathfrak{V}$ which contains the origin as an inner point. Let $F(x)$ be the associated Minkowski metric, $g_{ik}=\frac{1}{2}F^2(x)/\partial x^i\partial x^k$, where the symbol ∂ denotes the Fréchet derivative, and $g_{ik}(x)y^iy^k\geq m\|y\|^2$ for $y\neq 0$ and $m=m(x)>0$. Thus a Riemannian metric is defined except at the origin.

The curvature tensor in this metric is computed and vanishes when $\mathfrak{V}$ is two-dimensional. The only space of constant curvature, K, has $K=0$. Relationships between this metric and Finsler metrics are discussed.

Geodesics are defined by the usual differential equation. The system of geodesics coincides with the system of straight lines if and only if C is an ellipsoid having 0 as its center. *C. B. Allendoerfer* (Seattle, Nask.).

Putnam, C. R. On commutators and Jacobi matrices. Proc. Amer. Math. Soc. **7** (1956), 1026–1030.

Let A and B be bounded linear operators on a Hilbert space H and let A be normal. Let $C=AB-BA$ be the commutator of A and B and let W be the set in the complex plane consisting of the closure of the convex set (Cx, x), $x\in H$. Conditions are given on A which insure that 0 be an interior point of W, "interior" meaning interior to the segment W if W is one dimensional, $0=W$ if W is a single point and interior in the usual sense if W is two dimensional. Let A have the spectral resolution $A=\int z\,dK(z)$, K the spectral measure. Then if $\int_S dK=I$ where S is a set measurable K which for any $\varepsilon>0$ can be covered by a sequence of pairwise disjoint measurable sets the sum of whose diameters is less than ε, then 0 is interior to W. Simpler conditions will suffice if A is self adjoint or unitary. In particular, if A has a pure point spectrum then 0 is interior to W. It is shown as a corolarye that a large class of Jacobi matrices cannot have a plur point spectrum. *R. E. Fullerton.*

See also: Roelcke, p. 476; Wintner, p. 483; Lidskiĭ, p. 485; Harish-Chandra, p. 490; Gelfand et Šilov, p. 493.

TOPOLOGY

General Topology

Isbell, J. R.; and Rubin, Herman. Limit-preserving embeddings of partially ordered sets in directed sets. Proc. Amer. Math. Soc. **7** (1956), 812–813.

In a study of Alaoglu-Birkhoff convergence of a function f defined on a partially ordered set, the reviewer showed that directed systems were adequate for convergence by mapping A into the (directed) system T of its cofinal residual subsets. This is improved in this note as follows: Let B be the directed system of pairs (t, a), with t in T and a in t, with order defined by $(t, a)>(t', a')$ if and only if $t\subset t'$ or else $t=t'$ and $a>a'$. Let $\varphi(t, a)=a$, and let $\Phi(a)=(A, a)$; this embeds A into B in such a way that $F=f\varphi$ has precisely the same limit as f: more precisely, both φ and φ^{-1} carry cofinal residual sets to the same kind of sets. *M. M. Day* (Seattle, Wash.).

Gillman, Leonard. On a theorem of Mahlo concerning anti-homogeneous sets. Michigan Math. J. **3** (1955–1956), 173–177.

Novák asked if there exists a continuous antihomogeneous set [Colloq. Math. 3 (1955), 171, problem 134]. The author discusses some results of Mahlo [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. **63** (1911), 187–225] which show that if such a set exists it must be an enormously large inaccessible number. The author obtains some other necessary conditions for the existence of an antihomogeneous set.

(An ordered set is called continuous if it is dense and has no gaps; it is antihomogeneous if no two of its elements have the same character [see Hausdorff, Grundzüge der Mengenlehere, Veit, Leipzig, 1914]).

P. Erdős.

Burgess, C. E. A note on the separation of connected sets by finite sets. Proc. Amer. Math. Soc. **7** (1956), 1115–1116.

Let S be a topological space in the sense of Whyburn [Analytic topology, Amer. Math. Soc. Colloq Publ., v. 28, New York, 1942, p. 1; MR **4**, 86], that is, a T_1-space in the sense of Alexandroff and Hopf [Topologie, Bd. I, Springer, Berlin, 1935]. Suppose that S is connected, and contains more than one point. The author shows that if D is an open set such that every infinite subset of D contains a finite set that separates S, then some pair of points in D separates S. From this theorem it follows that if S is as above, and n is a positive integer such that S is separated by every set consisting of n points of S, then every open set contains a pair of points that separates S. These theorems are related to earlier results of Kline [Proc. Nat. Acad. Sci. U.S.A. **9** (1923), 7–12] and Whyburn [Monatsh. Math. Phys. **36** (1929), 305–314]. *E. E. Moise.*

Ghéorghiev, Ghéorghi Iv. Espaces topologiques et fonctions d'ensembles. Univ. d'Etat Varna "Kiril Slavianobâlgarski" Fac. Tech. Constructions. Annuaire **3** (1947–1948), 161–201 (1949). (Bulgarian. French summary)

Etude des espaces topologiques en prenant pour terme primitif la notion de l'intérieur d'un ensemble. {Remarque du Réf.: cf. A. Appert, Mathematica, Cluj **11** (1935), 229–246; là l'étude analogue est faite.}

Đ. Kurepa (Zagreb).

Ghéorghiev, Ghéorghi Iv. Espaces topologiques et fonctions d'ensembles. Univ. d'Etat Varna Fac. Tech. Méc. Annuaire **4** (1948–1949), 1–42. (Bulgarian. French summary)

Etude des espaces topologiques en prenant comme terme primitif la notion de bord X^b d'un ensemble X.

Un ensemble R est un espace si l'on a une transformation faisant correspondre à chaque $X \subseteq R$ un $X^b \subseteq R$ tel que $(X \cap Y)^b = X^b \cap Y \cup X \cap Y^b$, $(X \setminus X^b)^b = 0$, $R^b = 0$. On définit les G-ensembles (ouverts) comme solutions de l'équation $X^b = 0$; les complémentaires de G-ensembles sont les F-ensembles. On prouve plusieurs propositions; en particulier: X^b est la différence de X et de l'union de tous les ouverts $\subseteq X$, et dualement: X^b est la partie commune de X et des ensembles fermées $\supseteq R \setminus X$.

La b-topologie ainsi introduite est mise en relation en particulier avec la G-topologie et la F-topologie dans lesquelles les notions primitives sont: G-ensembles et F-ensembles respectivement. *Đ. Kurepa* (Zagreb).

McAuley, Louis F. Paracompactness and an example due to F. B. Jones. Proc. Amer. Math. Soc. **7** (1956), 1155–1156.

M. E. Rudin has reported an example of a separable normal space which is not paracompact [Bull. Amer. Math. Soc. **61** (1955), 573]. The author shows, by introducing a suitable topology, that a much earlier example due to F. B. Jones [ibid. **43** (1937), 671–677] furnishes an example of a space with the stated properties of Rudin's example. The definition of a paracompact space had not been stated at the time that Jones's paper was published; it was first given by J. Dieudonné [J. Math. Pures Appl. (9) **23** (1944), 65–76; MR **7**, 134]. *E. E. Moise.*

Newns, W. F. Uniform spaces with unique structure. Amer. J. Math. **79** (1957), 48–52.

L'auteur donne une démonstration plus simple d'un théorème de R. Doss [Amer. J. Math. **71** (1949), 19–23; MR **10**, 557] suivant lequel une condition nécessaire et suffisante pour qu'un espace complètement régulier E n'ait qu'une seule structure uniforme est que dans tout couple A, B d'ensembles fermés normalement séparés dans E (i.e. tels qu'il y ait une fonction numérique continue dans E, à valeurs dans [0, 1], égale à 0 dans A et à 1 dans B), l'un au moins est compact. Il prouve aussi que cette condition équivaut à la condition qu'il n'y ait sur E qu'une seule structure uniforme précompacte. Il discute aussi la façon dont se comporte la propriété d'avoir une seule structure uniforme relativement aux opérations usuelles de passage à un sous-espace, à un produit ou à un quotient. *J. Dieudonné* (Evanston, Ill.).

Allamigeon, André-Claude. Espaces homogènes symétriques harmoniques à groupe semi-simple. C. R. Acad. Sci. Paris **243** (1956), 121–123.

It is shown that a homogeneous symmetric space G/H, with G semi-simple, is harmonic if and only if its isotropy group acts transitively on every sphere of radius $\neq 0$.

W. Ambrose (Cambridge, Mass.).

Suvorov, G. D. Remarks on a theorem of Lavrent'ev. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 3–8. (Russian)

Let D be a bounded plane region containing 0 and let Z_i be points of D. Lavrent'ev [Dokl. Akad. Nauk SSSR (N.S.) **4** (1936), 207–209] defined a relative distance $\rho(Z_1, Z_2) = \min(\rho_1(Z_1, Z_2), \rho_2(Z_1, Z_2))$ where $\rho_1(Z_1, Z_2) =$ inf of length of polygonal paths in D connecting Z_1 with Z_2, and $\rho_2(Z_1, Z_2) =$ inf of length of polygonal paths cutting D into two connected regions such that Z_i both lie in that sub-region not containing zero. Lavrent'ev then considered Cauchy sequences in this metric and thus defined ideal boundary points of the region, but the function ρ extended to these Cauchy sequences has not the triangle inequality. The present paper replaces ρ by $\bar{\rho}$, the inf of the ρ-length of finite chains of points connecting the two desired points of the closure of D. It is proved that this has the triangle property and that

$$\bar{\rho}(Z_1, Z_2) = \inf[\rho(Z_1, Z) + \rho(Z, Z_2)],$$

where the inf is taken over Z in D. *M. M. Day.*

Wilder, R. L. Concerning a problem of Alexandroff. Michigan Math. J. **3** (1955–1956), 181–185.

Extending earlier work of Alexandroff [Ann. of Math. (2) **36** (1935), 1–35], the author shows that a locally compact Hausdorff space which is lc^n has no n-dimensional condensation at any point. Simple examples show that the converse fails for $n \geq 1$. However, the following theorem is proved: If a locally compact separable metric space is of finite dimension and has no n-dimensional condensation at any point for any n, then the space is locally connected (in dimension zero). The proof depends on the following result which is of independent interest: Let M be a closed connected subset of E^n such that 1) the complementary domains form a null sequence, and 2) the boundary of any complementary domain is regularly s-accessible from the domain for $s = 0, 1, \cdots, n-2$. Then M is locally connected. *E. G. Begle* (New Haven, Conn.).

Baum, John D. P-recurrence in topological dynamics. Proc. Amer. Math. Soc. **7** (1956), 1146–1154.

Let (X, T, π) denote a transformation group where X is a compact Hausdorff space and T is an abelian topo-

logical group. Let P denote a replete semigroup in T which is a proper subset of T. ECT is said to be P-extensive provided $E \cap pP \neq 0$ for each $p \in P$. If $x \in X$, x is said to be P-recurrent provided that for each neighborhood U of x there exists a P-extensive set E in T such that $r \in E$ implies $xr \in U$. If $x \in X$, x is said to be P-regionally recurrent provided that for each neighborhood U of x there exists a P-extensive set E in T such that $r \in E$ implies $Ur \cap U \neq 0$. The author is able to extend most of Chapter 7 of Gottschalk and Hedlund's "Topological dynamics" [Amer. Math. Soc. Colloq. Publ., v. 36, Providence, R.I., 1955; MR 17, 650] from recurrence to P-recurrence. The major results are the inheritance theorems for P-recurrence and P-regional recurrence and the corresponding theorems for the P-center of (X, T, π). In addition to these results the author secures relations between P-recurrence and incompressibility and shows that when $x \in X$ is P-recurrent, then x is compactly recurrent. It is shown that if A is the set of all P-regionally recurrent points of X, then A is nonvacuous, closed and invariant in X; hence, by an earlier theorem of the author [Trans. Amer. Math. Soc. 77 (1954), 506–519; MR 16, 503], if T is generative and separable, then each point of $X-A$ is P-asymptotic to A. *W. R. Utz.*

van den Dungen, Frans; Hontoy, Paul; et Janssens, Paul. Etude des oscillations à déferlement par les méthodes de l'analyse topologique. Vérification dans le cas d'un multivibrateur. C. R. Acad. Sci. Paris 243 (1956), 627–630.

Smirnov, Yu. M. On the dimension of proximity spaces. Mat. Sb. N.S. 38(80) (1956), 283–302. (Russian)

Detailed proofs are given of the results on the δ-dimension of proximity spaces contained in a previous note by the author [Dokl. Akad. Nauk SSSR (N.S.) 95 (1954), 717–720; MR 16, 845]. In the last section, the (covering) dimension of completely regular spaces (cf. loc. cit.) is considered. It is stated that the relation between the δ-dimension and the dimension of completely regular spaces is not so straightforward as supposed in the above note. Nevertheless, several important theorems are proved: $\dim R = \dim \beta R$; $\dim A \leq \dim R$ if $A \subset R$ is normally closed, $\dim R = \sup \dim A_i$ if $R = \bigcup A_i$, A_i are normally closed [cf. Katětov, Časopis Pěst. Mat. Fys. 75 (1950), 79–87; MR 12, 119]; $\dim R$ is the least n such that, for any normal finite covering α, there exists an α-mapping of R into (or: onto) an n-dimensional polyhedron; $\dim R$ is the least n such that, for any mapping f of R into an $(n+1)$-cell Q^{n+1} and any closed $A \subset R$ with $f(A) \subset S^n$ (boundary of Q^{n+1}) there exists a mapping f^* of R into S^n with $f = f^*$ on A. *M. Katětov* (Prague).

Smirnov, Yu. M. On metric dimension in the sense of P. S. Aleksandrov. Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956), 679–684. (Russian)

The metric dimension δmM of a metric space M is the least number r such that M has coverings of arbitrarily small mesh ε and of order $\leq r+1$. It is invariant under uniformly continuous homeomorphisms, and $\delta mM \leq \dim M$. An example of a space M with $\delta mM < \dim M$ was given by K. A. Sitnikov [Dokl. Akad. Nauk SSSR (N.S.) 88 (1953), 21–24; MR 14, 894]. It is shown that

$$\delta m(A \cup B) \leq \delta mA + \delta mB + 1$$

and, if $A \subset M$, $\delta mA \leq \delta mM$ and there is a G_δ-set H with $A \subset H \subset M$ such that $\delta mA = \delta mH$. For subsets M of euclidean space E^n, $\delta mM \leq r$ if and only if for each $\varepsilon > 0$ there is an ε-mapping into an r-dimensional locally finite polyhedron. The author asks if the polyhedron can always be chosen to lie in E^n. *C. H. Dowker* (London).

See also: Bagemihl, p. 456; Jaffard, p. 464; Leader, p. 470; Vinograd, p. 482; Garbar', p. 483; Wallace, p. 490; Matschinski, p. 502; Dantzig, p. 536; Kuhn, p. 546; Rashevsky, p. 548.

Algebraic Topology

Begle, E. G. The Vietoris mapping theorem for bicompact spaces. II. Michigan Math. J. 3 (1955–1956), 179–180.

In part I [Ann. of Math. (2) 51 (1950), 534–543; MR 11, 677], the author proved that if f is a mapping of X onto Y such that for each point y of Y, and for each integer k, $0 \leq k \leq n$, the Vietoris homology group (over a field or an elementary compact group as coefficient group) $H_v^k(f^{-1}(y)) = 0$, then the induced homomorphism $f^*: H_v^n(X) \to H_v^n(Y)$ is an isomorphism. In the present note, the author proves that the induced homomorphism $f^*: H_v^{n+1}(X) \to H_v^{n+1}(Y)$ is an epimorphism. An example is given that neither of the above results is correct when the coefficient group is taken to be the group of integers. *S. T. Hu* (Detroit, Mich.).

van der Waerden, B. L. Berichtigung und Ergänzung zur Arbeit "Die Cohomologietheorie der Polyeder", Math. Ann. Bd. 130, S. 87 (1955). Math. Ann. 132 (1956), 130–133.

This contains corrections and additions to an earlier paper of the author [Math. Ann. 130 (1955), 87–101; MR 17, 291]. *E. Spanier* (Chicago, Ill.).

Lashof, R. Classification of fibre bundles by the loop space of the base. Ann. of Math. (2) 64 (1956), 436–446.

If B is an arcwise connected topological space one can define a composition law in the space Ω of loops in B (with a given base point x_0). This can be done in a number of ways; in each case one has to sacrifice at least one of the group axioms. The author has chosen to define Ω as a groupoid. Two loops ω_1 and ω_2 are composed to form $\omega_1\omega_2$ only if the second half of ω_1 coincides with the first half of ω_2 taken in the opposite direction; the loop $\omega_1\omega_2$ is then formed by juxtaposing the first half of ω_1 with the second half of ω_2.

Using a definition of fibre bundle which is slightly more general than the usual one (the existence of local cross-sections is not assumed) the author is then able to prove the following main theorem. Let B be an arcwise and locally arcwise connected, metrisable space, then the equivalence classes of principal bundles with group G and base B are in 1-1 correspondence with the "conjugacy classes" of continuous homomorphisms of the groupoid of loops of B into G.

Applications are given to the case of totally disconnected groups, and to problems of reduction of structure group. In the remark on the final page there is a confusion between sheaves and groups, though this does not affect the conclusions.

Similar results, based on the use of simplicial paths in a simplicial complex, have been obtained by J. Milnor [Ann. of Math. (2) 63 (1956), 272–284; MR 17, 994]. *M. F. Atiyah* (Cambridge, England).

Bernard, Daniel. Sur l'intersection des sous-espaces fibrés principaux d'un espace fibré principal. C. R. Acad. Sci. Paris **243** (1956), 1714–1716.

Let $H(B, G, p)$ be a principal fibre bundle with base B, group G and projection p. Let G' be a closed subgroup of G. Then $H' \subset H$ is a G'-principal sub-bundle of H if it possesses a principal fibre bundle structure $H'(B', G', p')$ such that: 1) the topology of H' is that induced by the topology of H, 2) p' is the restriction of p to H', 3) the operation of G' on H' is the restriction of the operation of G on H. The author gives conditions on a pair of subgroups G', G'' of G under which any G'-principal sub-bundle of H and any G''-principal sub-bundle of H intersect in a $G' \cap G''$-principal sub-bundle of H. The case when G is a Lie group and the bundles are differentiable is also considered. Applications are promised in a forthcoming note. *M. F. Atiyah* (Cambridge, Mass.).

Moisil, Gr. C. Une interprétation du groupe fondamental d'une variété différentiable. Com. Acad. R. P. Romîne **5** (1955), 1407–1410. (Romanian. Russian and French summaries)

On a differentiable manifold let there be given linear differential forms ω_i^j, ω_i $(i, j=1, \cdots, r)$. Suppose the differential system $d\psi_i = \sum \omega_i^j \psi_j + \omega_i$ is completely integrable. When continued along a path issuing from a fixed point and ending at this point, the solution vector ψ_i is subject to a linear transformation depending on the path. The author observes that the group of such transformations is a representation of the fundamental group of the manifold. *W. Fenchel* (Copenhagen).

Boltyanskiĭ, V. G. Fibering of function spaces. Trudy Moskov. Mat. Obšč. **5** (1956), 299–309. (Russian)

Let S be a path-connected, locally contractible space and let the topological group G operate effectively and transitively on S. Let $q \in S$ and let Γ be the stability group of q. Suppose that the projection $p: G \to S$, given by $p(g)=gq$ gives G the structure of the total space of a fibre bundle (with local cross-section) with base S and fibre Γ. The author describes such a space S as an s-space (with respect to G).

Let X be a compact Hausdorff space and A a closed subspace. Let

$$P=(S,q)^{(X\times I, X\times 0 \cup A\times I)},\ C=(S,q)^{(X\times I, X\times 1 \cup A\times I)},$$

and let $B \subseteq (S, q)^{(X, A)}$ be the subspace of maps which are nullhomotopic rel A. As usual, P, C, B have the compact-open topology. Let $p: P \to B$ be given by $(p\xi)(x)=\xi(x, 1)$, and let $\mathfrak{G}$ be the space $(G, \Gamma, e)^{(X\times I, X\times 1, X\times 0 \cup A\times I)}$, e being the unity in G, and let $\mathfrak{G}$ be given the structure of a topological group induced by that in G. Then $\mathfrak{G}$ operates on C by $(\gamma \cdot \xi)(x, t)=\gamma(x, t) \cdot \xi(x, t)$, $\gamma \in \mathfrak{G}$, $\xi \in C$. The author's main theorem is that the projection $p: P \to B$ gives P the structure of the total space of a fibre bundle with base B, fibre C, and group $\mathfrak{G}$. The author actually exhibits an appropriate coordinate bundle.

The remaining results are contained in five lemmas. The first two assert elementary facts about the compact-open topology for which the author gives quite unnecessarily involved proofs. Lemma 3: If C is path-connected, it is an s-space relative to $\mathfrak{G}$. Lemma 4: If C is an s-space relative to $\mathfrak{G}$, then its universal cover $\bar{C}$ is an s-space, relative to $\bar{\mathfrak{G}}$, where $\bar{\mathfrak{G}}$ is the space of paths on $\mathfrak{G}$ issuing from its unit element, $\bar{\mathfrak{G}}$ being given the obvious structure of topological group. Lemma 5: Every connected manifold admits the structure of an s-space. The author acknowledges that this result is due to Fuller [Ann. of Math. (2) **59** (1954), 219–226; MR **15**, 642]. *P. J. Hilton.*

Milnor, John. On manifolds homeomorphic to the 7-sphere. Ann. of Math. (2) **64** (1956), 399–405.

Until the appearance of this paper by Milnor it was thought probable that a given topological manifold might possess at most one differentiable structure. Now it is seen that a manifold as simple as the seven sphere S^7 possesses several inequivalent differentiable structures.

Milnor starts out by considering the class of compact oriented differentiable 7-manifolds M^7 such that the third and fourth cohomology groups with integer coefficients $H^3(M^7)$ and $H^4(M^7)$, are zero. For every such manifold he defines an invariant $\lambda(M^7)$ which is an integer modulo 7. He then defines several differentiable structures on the 7-sphere and proves that his invariant λ is not the same for all of these structures.

The definition of the invariant $\lambda(M^7)$ rests on Thom's theory of "cobordisme" [Comment. Math. Helv. **28** (1954), 17–86; MR **15**, 890], where it is proved that every compact 7-manifold M^7 is the boundary of an 8-manifold B^8. Let $\mu \in H_7(M^7)$ be an orientation, and $\nu \in H_8(B^8, M^7)$ an orientation such that $\partial\nu=\mu$. Define $\tau(B^8)$ to be the index of the quadratic form on $H^4(B^8, M^7)$ given by the formula $\alpha \to \langle \nu, \alpha \rangle$. Let $p_1 \in H^4(B^8)$ be the first Pontrjagin class, and $\bar{p}_1 \in H^4(B^8, M^7)$ the unique class which maps into $p_1(H^3(M^7)=H^4(M^7)=0)$. Define a Pontrjagin number $q(B^8)$ to be $\langle \nu, \bar{p}_1^2 \rangle$.

Milnor's first theorem may now be stated. Theorem: The residue class of $2q(B^8)-\tau(B^8)$ modulo seven does not depend on the choice of the manifold B^8.

Having proved the preceding, he defines the invariant $\lambda(M^7)$ to be the residue class of $2q(B^8)-\tau(B^8)$, and it remains to consider examples of 7-manifolds. The examples in question are 3-sphere bundles over the 4-sphere with the rotation group SO(4) as structural group. Let R^4 be 4-dimensional euclidean space, and consider it as the space of quaternions. Define $f_{h,j}: S^3 \to \mathrm{SO}(4)$ by $f_{h,j}(u)v=u^h v u^j$, where the multiplication on the right is that of the quaternions. For each odd integer k, let M_k^7 be the total space of the 3-sphere bundle over S^4 determined by $f_{h,j}$, where $h+j=1$, $h-j=k$.

The following theorem is now proved.

Theorem: The manifold M_k^7 is homeomorphic with the seven sphere, and the invariant $\lambda(M_k^7)$ is the residue class module 7 of k^2-1, and consequently if $k^2-1 \not\equiv 0 \bmod 7$ the manifold M_k^7 carries a differentiable structure which is not equivalent to the usual one on the sphere S^7.

It is also proved by the author that either (a) there exists a closed topological 8-manifold which does not posses any differentiable structure, or (b) the Pontrjagin class p_1 of an open 8-manifold is not a topological invariant. *J. C. Moore* (Princeton, N.J.).

Schubert, Horst. Knoten mit zwei Brücken. Math. Z. **65** (1956), 133–170.

The author has previously considered [Math. Z. **61** (1954), 245–288; MR **17**, 292] the least integer b for which a given knot or link type has a regular projection with b overpasses. Only for the trivial knot is $b=1$; the present paper is concerned with the classification of the knots and links for which $b=2$. These have been previously considered [Bankwitz and Schumann, Abh. Math. Sem. Hamburg. Univ. **10** (1934), 263–284; Reidemeister, **11** (1935), 102–109] under the untranslatable name "Vier-

geflechte". For a knot, b is identical with the "crookedness" introduced by Milnor [Ann. of Math. (2) **52** (1950), 248–257; MR **12**, 273]; thus the knot types considered are just those whose total curvature is 4π.

If $b=2$ the knot or link has a "normal form" (which is not unique). This is a regular projection with two overpasses such that each of the two underpasses goes alternately under them. Each overpass contains the same number, say $\alpha-1$, of overcrossing points (so that the total number of crossings is $2\alpha-2$). These are numbered $1, 2, \cdots, \alpha-1$, in the order opposite to that induced by a selected orientation, and each index is prefixed by $+$ or $-$ according as the crossing is right- or left-handed. Then, as one travels along an underpass in the direction indicated by the orientation, the overpasses are gone under alternately, and the signed indices of these $\alpha-1$ crossings, in the order in which they are encountered belong to the residue classes $\beta, 2\beta, \cdots, (\alpha-1)\beta \pmod{2\alpha}$, where β is relatively prime to 2α. The normal form is completely described by the pair (α, β). The symbol $(0, 0)$ is associated with the trivial link and the symbol $(1,1)$ is associated with the trivial knot; otherwise $\alpha \geqq 2$ and β is a residue class modulo 2α prime to 2α. When α is odd there is only one component and we have a knot, when α is even the link has two components each of which is a trivial knot.

The principal theorem is that the knots or links represented by two normal forms (α, β) and (α', β') belong to the same semilinear oriented type in oriented 3-space if and only if $\alpha=\alpha'$ and $\beta' \equiv \beta^{\pm 1} \pmod{2\alpha}$. Reorientation of space changes (α, β) to $(\alpha, -\beta)$, and reorientation of the knot or link does not affect (α, β). Reorientation of one component of a link replaces (α, β) by $(\alpha, \beta+\alpha)$. It is shown that (α, β) is amphicheiral if and only if $\beta^2 \equiv -1 \pmod{2\alpha}$. From this it is shown that a conjecture of Seifert is correct: The trivial knot and the figure-eight are the only doubled knots that are amphicheiral.

The 2-sheeted covering of the 3-sphere that has the knot or link represented by (α, β) as branch curve is the lens space (α, β) (where here β is a residue class modulo α). From the known classification of lens spaces [Reidemeister, Abh. Math. Sem. Hamburg. Univ. **11** (1935), 102–109; Moise, Ann. of Math. **56** (1952), 96–114; MR **14**, 72; Fox, ibid. **57** (1953), 547–560; MR **14**, 843] it follows that the author's semilinear classification of knots and links with two overpasses is also topological up to certain links whose linking number is zero. *R. H. Fox.*

See also: Nomizu, p. 489; Kobayashi, p. 503; Auslander, p. 507; Hirzebruch, p. 509; Wu, p. 536; Kuhn, p. 546.

GEOMETRY

Geometries, Euclidean and other

Fulton, Curtis M. A theorem on parallels. Proc. Amer. Math. Soc. **7** (1956), 1107–1108.

The author gives a non-metrical proof that a continuous "descriptive" plane (in the terminology of Russell and Whitehead) admits non-intersecting lines. He shows by an example that, without continuity, it may happen that every two lines intersect. The treatment does not differ essentially from that of Veblen [Trans. Amer. Math. Soc. **5** (1904), 343–384, pp. 348, 369–370; see also Coxeter Non-Euclidean geometry, Univ. of Toronto Press, 1942, p. 175; MR **4**, 50]. *H. S. M. Coxeter.*

Bielecki, A. Réduction des axiomes de congruence de Hilbert. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 321–324.

The author modifies the five axioms of congruence (known to be not independent) of Hilbert's Grundlagen der Geometrie [8th ed., Teubner, Stuttgart, 1956; MR **18**, 227] in the following manner: Axioms III_1, III_2, III_5 are retained, axiom III_3 is discarded, and axiom III_4 is replaced by $III_4{}^*$. If $\sphericalangle ABC$ is a given angle and α is a given half-plane bounded by the line KL $(K \neq L)$, there is at least one point N of α such that $\sphericalangle ABC \cong \sphericalangle KLN$. $III_4{}^{**}$. If $\sphericalangle ABC$ is a given angle and α a given half-plane bounded by the line KL $(K \neq L)$, there is at most one point N of α such that $\sphericalangle ABC \cong \sphericalangle KLN$. It is shown by a table of models that each of the axioms III_1, III_2, $III_4{}^*$, $III_4{}^{**}$, III_5 is independent of the others and of the preceding axioms (Incidence, Order). The details of the proof that the new system is equivalent to the Hilbert system are promised in another publication.

L. M. Blumenthal (Columbia, Mo.).

Kalitzin, Nikola Stilianov. Grundlagen der pseudoeuklidischen Geometrie. Univ. d'Etat Varna Fac. Tech. Méc. Annuaire **4** (1948–1949), 43–106. (Bulgarian. German summary)

Carver, Walter B. The conjugate coordinate system for plane euclidean geometry. Amer. Math. Monthly **63** (1956), no. 9, part II, iii+86 pp.

Disclaiming any originality in the method presented or the geometric contents of his work, the author undertakes to exhibit, in an elementary way, the advantages of the use of cyclic coordinates in connection with certain types of problems in plane geometry. The method has been used with success by many writers and was pioneered in this country by Frank Morley, both in his writings and in his lectures.

The author brings to his task his vast experience of many years as a teacher, writer, and editor which he puts to good use. His skill and competence are apparent on every page. Of his readers he prefers to expect "an interest in mathematics" rather than erudition, of which a first course in the calculus will suffice. The author thinks it more expedient to present in detail something the reader is likely to know already than to startle him by an argument based on premises which are foreign to him. And when the author has to make use of information which he cannot conveniently supply, he takes care to provide a precise reference to an accessible and readable source. The bibliography, consisting of forty-one items, at the end of the book refers to articles, in English, which appeared in journals published in the U.S., more than half of them in the American Mathematical Monthly.

If a member of a college or university faculty would want to follow in the author's footsteps and offer an introductory course in the use of cyclic coordinates, he is sure to lighten his own burden by putting this book into the hands of his students, and he also stands a good chance of strengthening their "interest in mathematics" as well.

The following list of chapter headings may convey an idea of the contents of the book. 1. Complex numbers and their graphical representation. 2. The conjugate coordinate system. 3. The straight line. 4. The circle. 5. Map equations and the derivative $D_t x$. 6. Symmetric functions

of turns. 7. Some geometry of the triangle. 8. The parabola. 9. Hypocycloids. 10. Epicycloids and trochoids. 11. The rectangular hyperbola. *N. A. Court.*

Marmion, A. Epi- ou hypocycloïdes semblables tangentes à 2, 3 ou 4 droites. Mathesis **65** (1956), 234–252.

Si les points M et N se déplacent sur la circumférence d'un cercle de façon que les arcs décrits ont la proportion k, la droite MN enveloppe une épi- ou hypocycloïde. En partant de cette définition l'auteur étudie les courbes tangentes à des droites données, surtout en considérant des valeurs rationnelles de k. *O. Bottema* (Delft).

Goormaghtigh, R. Sur une généralisation d'un théorème de Carnot. Mathesis **65** (1956), 192–196.

Il s'agit du théorème de Carnot relatif aux segments déterminés par une droite sur les côtés d'un polygone. L'auteur considère deux polygones $A_1A_2\cdots A_n$ et $A_1'A_2'\cdots A_n'$; A_iA_{i+1} et $A_i'A_{i+1}'$ se coupent au point $B_{i,i+1}$; alors

$$\prod \frac{A_iB_{i,i+1}}{B_{i,i+1}A_{i+1}} = \prod \frac{\sin \theta_{i,i+1}}{\sin \theta_{i+1,i}},$$

où $\theta_{i,i+1}$ et $\theta_{i,i-1}$ sont les angles de $A_i'A_{i+1}'$ et $A_i'A_{i-1}$ avec $A_i'A_i$. *O. Bottema* (Delft).

Pavlovič, S. V. Sur deux théorèmes de D. Pompeiu. Mathesis **65** (1956), 211–214.

Il s'agit des théorèmes: 1. Avec les distances aux sommets d'un triangle équilatéral d'un point quelconque du plan on peut construire un triangle; 2. si les côtés d'un polygone sont égaux, avec les distances de leurs milieux à un point quelconque du plan on peut construire un polygone. L'inversion de (1) est toujours possible; il est impossible d'invertir (2) pour un polygone régulier dont le nombre de côtés dépasse 3. *O. Bottema* (Delft).

Pozzolo Ferraris, Giulia. Applicazione alle quadriche del principio di Cavalieri sui volumi. Period. Mat. (4) **34** (1956), 113–121.

Radu, N. Imaginary foci of a conic. Gaz. Mat. Fiz. Ser. A. **8** (1956), 411–419. (Romanian)

Ionescu, D. V. On Stewart's relation. Gaz. Mat. Fiz. Ser. A. **8** (1956), 407–411. (Romanian)

Peterfi, I. On the definition of foci of a conic. Gaz. Mat. Fiz. Ser. A. **8** (1956), 406–407. (Romanian. Russian and French summaries)

Deaux, R. Sur les trièdres. Mathesis **65** (1956), 407–411.

Gurland, John. On Wallis' formula. Amer. Math. Monthly **63** (1956), 643–645.

It is shown how a basic theorem in mathematical statistics concerning unbiased estimators with minimum variance (the information inequality) can be used to prove an inequality for π, implying Wallis' formula. *E. L. Lehmann* (Berkeley, Calif.).

★ **Enriques, Federico. Los Elementos de Euclides y la critica antigua y moderna. Libros I-IV. [The elements of Euclid and ancient and modern criticism. Books I-IV.] Translation from the Italian edition by Jose Mingot Shelly.** Publicaciones del Instituto "Jorge Juan" de Matemáticas, Madrid, 1954. 216 pp.

The original Italian edition was published by Stock, Roma, 1925.

Freitag, Herta Taussig; and Freitag, Arthur H. Neo-pythagorean triangles. Scripta Math. **22** (1956), 122–131.

This paper is an elaboration of a note of the same title by E. Kasner [Scripta Math. **13** (1947), 43–47; MR 9, 299]. The six coordinates (x_i, y_i) of the vertices of a triangle are general complex numbers. Using trigonometric functions of complex argument and defining length by $d^2=(x_i-x_j)^2+(y_i-y_j)^2$, one will ordinarily not be led into logical contradiction if one defines all elements of the triangle (area, sides, angles, radii of inscribed and circumscribed circles, medians, etc.) by applying the formulae of plane trigonometry and analytic geometry. Characteristic for neo-pythagorean triangles is the relation $a^2+b^2+c^2=0$ between the sides. From the symmetry of this definition follows that the new formulae will possess properties of symmetry not present in the corresponding formulae for ordinary right triangles ($a^2+b^2-c^2=0$). A number of such relations are added to those given by Kasner.

Neither Kasner not the present authors consider the difficult question of construction (geometric representability) of their triangles. The authors of the paper under consideration seem to admit as trivial the case when one side lies in a minimal line (leading to a triangle with a side 0, but area a complex number $\neq 0$; etc.). Kasner had excluded this case from consideration. *A. Kempner.*

Court, N. A. Some missing theorems on the anticomplementary tetrahedron. Amer. Math. Monthly **63** (1956), 714–716.

The author gives some new results in affine solid geometry. A triangle ABC yields an "anticomplementary" triangle whose sides are the lines through A, B, C parallel to BC, CA, AB. The anticomplementary tetrahedron (T'') of a given tetrahedron (T) is defined analogously. The author proves that an edge of (T'') is trisected by the two faces of (T) which that edge meets. Also the twelve vertices of the four anticomplementary triangles of the four faces of (T) lie in pairs on the six edges of (T''). *H. S. M. Coxeter* (Toronto, Ont.).

Monseau, M. Sur une généralisation de la théorie des sphères podaires. Mathesis **65** (1956), 223–230.

Si (P, Q) est un couple de points isogonaux d'un tétraèdre on définit la sphère S_n concentrique à la sphère podaire S_1 et telle que les puissances de P par rapport à S_n et S_1 soient dans le rapport n. S_0 est la sphère de diamètre PQ, S_2 c'est la sphère orthoptique pour les méridiens de la quadrique de révolution inscrite, de foyers P, Q etc. L'auteur donne une série de théorèmes pour S_n; nous citons: le lieu des points tels que S_n est orthogonale à une sphère donné est une surface du quatrième ordre. *O. Bottema* (Delft).

Cavallaro, Vincenzo G. Note sulla geometria del triangolo ellissi d'ordine potenziale m, m—2-punto potenziale P_4 etc. Giorn. Mat. Battaglini (5) **4(84)** (1956), 248–260.

Goormaghtigh, R. Sur des ellipses associées aux droites de Simson d'un triangle. Mathesis **65** (1956), 395–401.

If a point M describes a circle, center O, circumscribed about a triangle T, the symmetric of the midpoint of OM with respect to the Simson line of M for T describes an ellipse E. The center of E coincides with the nine point center O_9 of T, etc.

The proposition is generalized by replacing the ordinary

Simson line by the Simson line of angle θ, as defined by C. Servais [Mathesis **46** (1932), 181–183]. The above proposition is modified in that the role of the midpoint of OM is now played by the point N of the mediator of OM at which OM subtends an angle equal to 2θ. The ellipse E_θ thus obtained has for its center the point of the mediator of the segment OH (H the orthocenter of T) at which the segment OH subtends an angle equal to 2θ, etc.

As the angle θ varies the ellipse E_θ envelops a conic Σ having O_θ for its center, etc.

The treatment is analytic and complex coordinates are used. *N. A. Court* (Norman, Okla.).

Hansen, Walter. Ein Beweis der Ptolemäischen Ungleichung. Arch. Math. **7** (1956), 320–322.

The author uses quaternions to prove the classical formula

$$p+q \geqq r,$$

where p, q, r are the three products of the three pairs of opposite edges of a tetrahedron. It is pointed out that the case of equality occurs only if the four given points are the vertices of a cyclic quadrilateral, or when some of them or all of them coincide. *N. A. Court.*

Deaux, R. Sur le premier point de Lemoine d'un tétraèdre. Mathesis **65** (1956), 411–412.

Given a tetrahedron (A), any point P and the centroid P' of the pedal tetrahedron of P for (A) correspond to each other in an affinity ω having for proper double planes the planes of symmetry of the quadric surface Σ inscribed in (A) at the feet of the altitudes.

The center of this quadric is the first Lemoine point K of (A).

The author proves this proposition analytically and points out several corollaries. *N. A. Court.*

★**Kijne, Dono. Plane construction field theory.** Van Gorcum and Comp. N.V.-G. A. Hak and Dr. H. J. Prakke, Assen, Netherlands, 1956. 119 pp.

This thesis deals with the theory of elementary constructions (ruler, compasses, "right square") in plane euclidean geometry. Author gives a careful analysis of the concepts made use of which are certainly not always considered in this matter. Interesting is the introduction of decision operations, giving a decisive answer with regard to certain relations existing between some objects; e.g., if A and B are two points it must be decidable whether they are different or not; only if $A \neq B$ is provable the construction of the line through A and B has a definite sense. Beside the active and decision operations the author introduces selection operations (saying that it is possible to take "arbitrary" elements out of a set of equivalent objects) and adjungation operations. Special care is given to cases where a certain operation (e.g. the intersection of two circles) produces two objects, equivalent with respect to the preceding construction, but non-equivalent with respect to the given construction problem; Tietze's work [S.-B. Akad. Wiss. Wien. IIa. **118** (1909), 735–757] is referred to. Important is the concept of construction field, the set of objects in a certain geometry constructible from a basis; automorphisms of the field are considered. A last chapter is devoted to the Mohr-Mascheroni constructions and their relation to Archimedes' axiom. {The thesis is an interesting contribution to abstract construction theory.} *O. Bottema.*

Petrov, G. Sur les conditions nécessaires et suffisantes de l'existence du triangle. Univ. d'Etat Staline Fac. Tech. Constructions. Annuaire **5** (1949–1950), 127–152. (Bulgarian. French summary)

Given any three linear elements of a triangle, say, the sides b, c and the internal bisector l_a of the included angle, the author considers the often neglected question: what equalities and inequalities do these segments have to satisfy in order that the construction of the triangle shall be possible?

His method of attack consists in bringing in a fourth linear element, say, the third side, a, of the triangle and considering the homogeneous relation existing among the four segments, in the present case the known relation:

$$l_a^2(b+c)^2 = bc[(b+c)^2 - a^2].$$

If the four segments be considered as the rectangular homogeneous coordinates of a point

$$a=x,\ b=t,\ c=z,\ l_a=y$$

in Euclidean space of three dimensions, the above equality may be written as the equation of a surface:

$$y^2(z+t)^2 - zt(z+t)^2 - x^2 = 0.$$

The author discusses this surface, and particularly its traces in the coordinate planes, in connection with the inequalities:

$$x+z-t>0,\ x-z+t>0,\ -x+z+t>0,$$

which express the basic relations between the sides of the triangle. This discussion is restricted to the first octant, due to the nature of the problem.

The author arrives at the following necessary and sufficient condition for the constructibility of the triangle:

$$0 < l_a < 2bc/(b+c),$$

an elegant relation for which this reviewer knows no explicit reference. [Notice that the right hand side of this inequality is the harmonic mean of b and c].

The method is general. The author applies it to a considerable number of cases, involving the sides, the altitudes, the medians, the internal bisectors of the angles, the inradius, and the circumradius. Some of the conditions obtained are so involved as to be of little interest, and a good many others are practically obvious from the corresponding figure in the plane.

The problems which are considered lead to one or two solutions. *N. A. Court* (Norman, Okla.).

Lenz, Hanfried. Über die Konstruierbarkeit der Schnittpunkte dreier Flächen zweiter Ordnung. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1955**, 45–51 (1956).

★**[Anspach, Pierre A. L.] Aperçu de la théorie des polygones reguliers. III.** Bruxelles, 1955, pp. 193–298.

The first two parts of this book were reviewed last year [MR **17**, 397, 1122]. This third part seems to be the concluding one judging by the last paragraph (574 "Coda"), where the author bids his readers farewell.

Here is the table of contents: 1. Transversal of the heptal triangle, pp. 193–247. 2. New theorems on the heptal lemniscate, pp. 247–255. 3. Conics used in graphical determinations. Six conics connected with the regular enneagon, pp. 255–278. 4. Conics connected with a regular heptagon, with a regular decatregon, pp. 279–298.

The book includes seven full page figures. No bibliography, no index. *N. A. Court* (Norman, Okla.).

Matschinski, Matthias. Formule d'Euler généralisée et condition de son application. Courbure extérieure et intérieure. C. R. Acad. Sci. Paris **243** (1956), 1595–1598.

The author continues his work on regular skew polyhedra [same C. R. **243** (1956), 472–475; MR **18**, 227], distinguishing between finite and infinite polyhedra by means of the Euler-Poincaré characteristic and the concepts of interior and exterior curvature.

H. S. M. Coxeter (Toronto, Ont.).

Rogov, V. Ya. Classification of arbitrary plane figures in a square grid. Voronež. Gos. Univ. Trudy Fiz.-Mat. Sb. **27** (1954), 47–74. (Russian)

Al-Dhahir, M. W. A class of configurations and the commutativity of multiplication. Math. Gaz. **40** (1956), 241–245.

The author describes the connections between a pair of Möbius tetrahedra, Veblen's two-quadrangles theorem, Baker's eight-lines theorem, and commutativity, in the manner of pp. 444–445 of Coxeter, Bull. Amer. Math. Soc. **56** (1950), 413–455 [MR **12**, 350]. He draws attention to some relevant remarks by Reidemeister [Jbr. Deutsch. Math. Verein. **38** (1929), Abt. 2, 71] and Schönhardt [ibid. **40** (1931), Abt. 2, 48–50]. He gives an easy method for constructing a model of the Möbius tetrahedra by using threads to connect appropriate vertices of two quadrangles drawn on a piece of folded cardboard.

H. S. M. Coxeter (Toronto, Ont.).

Norden, A. P. On self-adjoint forms of biaxial space. Kazan. Gos. Univ. Uč. Zap. **114**, no. 2 (1954), 3–12. (Russian)

A biaxial space of the elliptic type is a three-space with fundamental group isomorphic with that subgroup of the group of projective transformations by which a spatial involution of matrix $g^{\alpha\beta}$, $\alpha, \beta=1, 2, 3, 4$ is invariant; the characteristic numbers of $g^{\alpha\beta}$ being i, i, $-i$, $-i$, $g_\lambda{}^\alpha g_\beta{}^\lambda=-\delta_\beta{}^\alpha$, $g_\alpha{}^\alpha=0$. We thus have transformations $\tilde{x}^\alpha=g_\beta{}^\alpha x^\beta$, $\tilde{\xi}_\alpha=g_\alpha{}^\beta \xi_\beta$. Lines connecting corresponding points are called singular and form an elliptic line congruence, which also contains the lines common to corresponding planes, and of which the axes are two intersecting conjugate imaginary lines, the absolute lines. [See further Norden, Mat. Sb. N.S. **24(66)** (1949), 429–455; MR **11**, 135.] The linear complexes containing this congruence form the absolute pencil of complexes, the special complexes in this congruence contain the absolute lines. This allows the establishment in this pencil of an elliptic metric. Spheroids are ruled quadratic surfaces formed by rulings belonging to the singular lines. The inner geometry of a real plane is euclidean, the improper line being the singular line connecting the points in which the absolute lines intersect that plane. —. Now it is possible to study such biaxial spaces in which there exists a self conjugate tensor, that is, a tensor for which $\tilde{a}_{\alpha\beta}=g_\alpha{}^\lambda g_\beta{}^\mu a_{\lambda\mu}=ka_{\alpha\beta}$; k can only be ± 1. Investigated are the cases in which $a_{\alpha\beta}$ is either symmetrical or alternating, so that there are four cases. In the first case, $\tilde{a}_{\alpha\beta}=a_{\alpha\beta}$, $a_{\alpha\beta}$ symmetrical, the tensor $b_{\alpha\beta}=g_\alpha{}^\lambda a_{\lambda\beta}$ is alternating, and $a_{\alpha\beta}x^\alpha x^\beta=0$ represents a self-conjugate spheroid. In the second case, $\tilde{a}_{\alpha\beta}=-a_{\alpha\beta}$, $a_{\alpha\beta}$ symmetrical, both $a_{\alpha\beta}x^\alpha x^\beta=0$ and $b_{\alpha\beta}x^\alpha x^\beta=0$ represent surfaces, called bicylinders, that is, surfaces with rulings belonging to two mutually orthogonal complexes of the absolute pencil. In the third case, $\tilde{a}_{\alpha\beta}=-a_{\alpha\beta}$, $a_{\alpha\beta}$ skew, $a^{\alpha\beta}b_{\alpha\beta}=0$, both $a^{\alpha\beta}$ and $b_{\alpha\beta}$ determine two mutually orthogonal complexes of the absolute pencil, and in the fourth case, $\tilde{a}_{\alpha\beta}=a_{\alpha\beta}$, $a_{\alpha\beta}$ skew, the $b_{\alpha\beta}$ belongs to the first case, and the $a_{\alpha\beta}$ determines a so-called cyclical complex, in involution with any complex of the absolute pencil. The metrical properties of these four spaces are described in more detail.

D. J. Struik (Cambridge, Mass.).

Pickert, Günter. Projektive Ebenen über Neokörpern. Wiss. Z. Friedrich-Schiller-Univ. Jena **5** (1955/56), 131–135.

We say that a system S is a planar neofield if there is a projective plane which can be coordinatized by S in such a way that lines are given by linear equations and furthermore in S multiplication of non-zero elements is a group G, and both distributive laws hold. The addition in S will be a loop, but need have no further properties. It is shown that a plane is given by a planar neofield if and only if there is a triangle ABC such that the theorem of Desargues is valid in all cases with any one of the vertices as center and the opposite side as axis. It is further shown that if for any other center and axis the theorem of Desargues is valid, then the plane is Desarguesian and S is a division ring. As a consequence of these results, every collineation of the plane may be obtained by permuting A, B, C, multiplying coordinates, and applying automorphisms of G.

Marshall Hall, Jr. (Columbus, Ohio).

Popescu, M. Certain theorems on quadrics. Gaz. Mat. Fiz. Ser. A. **8** (1956), 396–399. (Romanian. Russian and French summaries)

Ewald, Günther. Axiomatischer Aufbau der Kreisgeometrie. Math. Ann. **131** (1956), 354–371.

Circle geometry is developed on the basis of the following axioms relating the undefined concepts of points $p, q, \cdots$, circles $C_1, C', \cdots$, incidence ε, orthogonal $\perp$: C and $p \notin C$ exist. There is exactly one C through three distinct points. Every circle contains at least three points. If $p \in C$ and $q \notin C$, then exactly one C' containing p and q and intersecting (touching) C only at p exists. If $C \perp C'$, then $C' \perp C$. If $C' \perp C$, then C and C' intersect. A circle orthogonal to two touching circles passes through the point of contact. Two circles orthogonal to two non-intersecting circles intersect. If $p \in C$ and $q \neq p$, then exactly one $C' \perp C$ through p and q exists. For distinct p, q, r there is exactly one C through r orthogonal to all circles through p and q. Finally, the following restricted form of the "pencil theorem" is postulated: Let eight distinct points be grouped into four pairs $P_1, \cdots, P_4$. If five of the six quadruples $P_i \cup P_j$ lie on five distinct circles C_k ($k=1, \cdots, 5$), and no C orthogonal to three C_k through the same P_i exists, then the remaining $P_i \cup P_j$ lies on a circle.

It is shown that a point and line can be defined as identified with bundles and pencils of circles to satisfy the projective axioms, and that the system of points and circles can be imbedded in P^2 over a skew field such that the circles are represented by generalized tetracyclical coordinates. *H. Busemann* (Los Angeles, Calif.).

Petrow, Georgi. Die Projektionsmethode von Monge im hyperbolischen Raume. Univ. d'Etat Varna "Kiril Slavianobălgarski" Fac. Tech. Méc. Annuaire 3 (1947–1948), 67–84 (1949). (Bulgarian. German summary)

Hazanov, M. B. On the application of a new theory of areas in the Lobačevskiĭ plane to the derivation of the curvature formula of a curve. Kabardinskiĭ Gos. Ped. Inst. Uč. Zap. 8 (1955), 21–24. (Russian)

Kobayashi, Shoshichi. Holonomy groups of hypersurfaces. Nagoya Math. J. 10 (1956), 9–14.

It is shown that the restricted homogeneous holonomy group of an n-dimensional compact manifold which can be isometrically imbedded in R^{n+1} is the full orthogonal group. *W. Ambrose* (Cambridge, Mass.).

See also: Lemmon, p. 461; Linnik, p. 467; Maschler, p. 473; Derry, p. 503; Svoboda, p. 504; Auslander, p. 507.

Convex Domains, Integral Geometry

Derry, Douglas. Convex hulls of simple space curves. Canad. J. Math. 8 (1956), 383–388.

It is well-known that the convex hull $\{M\}$ of a point set M of Euclidean n-space E^n is the union of simplexes of dimensions $\leqq n$ with vertices belonging to M. For particular classes of sets simplexes of lower dimensions suffice. The present paper deals with the case of a simple arc or closed curve A_n of order n in E^n, that is, an 1-1 continuous map of a closed segment or a circle into E^n with the property that no hyperplane of E^n contains more than n points of A_n. It is shown that every interior point P of the convex hull $\{A_n\}$ is in the interior of a simplex whose vertices are on A_n and whose dimension is $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd. For n odd this simplex is uniquely determined by P, while for n even there is a one-parameter family of such simplexes containing P. In the case of a curve A_n, n even, the boundary of $\{A_n\}$ is the union of all (closed) $(\frac{1}{2}n-1)$-simplexes with vertices on A_n; in the case of an arc $\frac{1}{2}n$-simplexes, two of whose vertices coincide with the end points of A_n, have to be added. A similar result holds for n odd. Finally it is pointed out that, for n even, the convex hull $\{A_n\}$ admits of a projectively invariant characterization: Every interior point of $\{A_n\}$ is the intersection of two $\frac{1}{2}n$-dimensional subspaces intersecting A_n in two sets of $\frac{1}{2}n+1$ points each which alternate on A_n. Some special cases of the results of the paper have been obtained previously, namely the case $n=3$ by Egerváry [Publ. Math. Debrecen 1 (1949), 65–70; MR 12, 46] and for certain algebraic A_n by Carathéodory [Rend. Circ. Mat. Palermo 32 (1911), 193–217] and by Karlin and Shapley [Mem. Amer. Math. Soc. no 12 (1953); MR 15, 512]. *W. Fenchel.*

Minagawa, Takizo. Remarks on the infinitesimal rigidity of closed convex surfaces. Kōdai Math. Sem. Rep. 8 (1956), 41–48.

Efimov noticed [Uspehi Mat. Nauk (N.S.) 3 (1948), no. 2(24), 47–158; MR 10, 324] that a convex surface of a certain class is infinitesimally rigid after omitting its planar portions, even if it contains parts where the curvature vanishes. This theorem is proved here under weaker hypotheses: the surface is of class C' and piecewise of class C'', the infinitesimal deformation vector is of class C'. *H. Busemann* (Los Angeles, Calif.).

Lumer, Gunter. Sets with connected spherical section. Soc. Parana. Mat. Anuário 2 (1955), 12–17. (Portuguese)

Let S be a closed surface in E_3. If the intersection of S with any spherical surface is always connected, then S itself is a spherical surface. *L. A. Santaló.*

Hadwiger, H. Integralsätze im Konvexring. Abh. Math. Sem. Univ. Hamburg 20 (1956), 136–154.

The bodies A of the k-dimensional euclidean space E_k which are unions of a finite number of convex bodies form the "convex ring" K. Let $\varphi(A)$, $A \in K$, be a functional which is additive and satisfies certain conditions of continuity and smallness at infinity. Denoting by K_r the sphere with center at the origin and radius r and by dE_ν the density for ν-dimensional linear spaces in E_k in the sense of integral geometry, the author defines

$$\varphi(E_\nu)=\lim_{r\to\infty}\varphi(E_\nu\cap K_r),\quad W_\nu(\varphi)=\frac{1}{c_{\nu k}}\int\varphi(E_\nu)dE_\nu,$$

where the last integral is extended over all E_ν in E_k, and proves the very general integral formula

$$\frac{1}{c_k}\int\varphi(A)dA=\sum_0^k\binom{k}{\nu}W_\nu(\varphi)W_{k-\nu}(A),$$

where dA denotes the cinematic density in E_k and the integral is extended over all bodies congruent to A; $W_{k-\nu}(A)$ are the ordinary Minkowski functionals and the constants c_k and $c_{\nu k}$ have the values

$$c_k=\frac{k!}{2}\,\omega_1\omega_2\cdots\omega_{k-1},\quad c_{\nu k}=\binom{k}{\nu}\frac{\omega_{k-1}\cdots\omega_{k-\nu}}{\omega_1\cdots\omega_\nu},$$

where ω_i denotes the volume of the i-dimensional unit sphere. This important formula, which generalizes previous results of the author [Arch. Math. 3 (1952), 470–478; 4 (1953), 374–379; MR 14, 1114; 15, 551] covers many particular cases. *L. A. Santaló* (Buenos Aires).

See also: Pták, p. 491; Laugwitz and Lorch, p. 495.

Differential Geometry

Fleming, W. H.; and Young, L. C. Representations of generalized surfaces as mixtures. Rend. Circ. Mat. Palermo (2) 5 (1956), 117–144.

This paper contains important contributions to the theory of generalized surfaces initiated by L. C. Young. In view of the large number of the definitions involved and of the complexity of the results we restrict ourselves to a description of two of the main theorems derived in the paper under review. Let R^3, V^3 denote Euclidean three-space and the unit sphere in Euclidean three-space respectively, and let $\Sigma=R^3\times V^3$. A generic point of Σ is then of the form (x, J), where $x\in R^3$ and $J=(j^1, j^2, j^3)$ with $|J|=1$. F is the space of all continuous functions $f(x, J)$ on Σ. Each f is extended to $R^3\times R^3$ by the agreement $f(x, J)=|J|f(x, |J|^{-1}J)$ for $J\neq 0$, $f(x, 0)=0$. The function f which is identically equal to 1 on Σ is denoted by f_a, while Φ^i, $i=1, 2, 3$, is defined by $\Phi^i(x, J)=j^i$. A generalized surface L is a linear functional on the space F. The norm $a(L)$ of L is defined by $a(L)=L(f_a)$. In stating further definitions, we refer the reader for details not given here to a paper by W. H. Fleming and L. C. Young [Trans. Amer. Math. Soc. 76 (1954), 457–484; MR 16, 721]. The g-boundary (generalized boundary) $\lambda(L)$ of a generalized surface L is defined as the restriction of L to exact functions $f\in F$, and Λ_0 is the class of all admissible g-boundaries. To each generalized surface L there corresponds a unique measure γ on Σ such that $L(f)=\int f d\gamma$ for

every $f \in F$. If there exists a point $x \in R^3$ such that $\gamma|(R^3-x)\times V^3|=0$, then L is termed a microsurface situated at x. Given a generalized surface L, one has a representation

$$L(f)=\int_{R^3} M_x(f)d\mu,$$

for every $f \in F$, where M_x is a microsurface of unit norm situated at x and μ is the measure defined over R^3 for γ-measurable sets by $\mu(A)=\gamma(A\times V^3)$. For $x \in R^3$ one defines then a vector $J(x)$ by the formula

$$J(x)=[M_x(\Phi^1), M_x(\Phi^2), M_x(\Phi^3)].$$

The track tL of L is the generalized surface defined for $f \in F$ by

$$(tL)(f)=\int_{R^3} f|x, J(x)|d\mu.$$

A generalized surface L is termed closed if $\lambda(L)=0$ and singular if $tL=0$. A generalized surface L is said to be situated in a set $A \subset R^3$ if $\gamma|(R^3-A)\times V^3|=0$. Let G be the class of all generalized surfaces, and for $R>0$ let G_R be the class of all those generalized surfaces which are situated in the cube with center at the origin, side-length R, and sides parallel to the coordinate axes. Let α_0 be a measure on Borel sets in G_R, α its completion in G_R, and let α be extended to subsets Ω of G by the agreement $\alpha(\Omega)=\alpha(\Omega\times G_R)$. A generalized surface L_0 is termed a mixture of a set Γ with measure α if Γ is α-measurable with $\alpha(\Gamma)=\alpha(G)<\infty$, α vanishes outside Γ, and

$$L_0(f)=\int_G L(f)d\alpha$$

for all $f \in F$. The quantity $\alpha(G)$ is termed the mass. Given $\lambda \in \Lambda_0$, the class of all sequential limits of Dirichlet surfaces with g-boundary λ is denoted by $E(\lambda)$. The symbol I denotes the class of all generalized surfaces L such that $a(L) \geqq 1$, and there exists a positive integer N and a sequence P_n of irreducible closed polyhedra such that $L=N \lim P_n$.

Two of the main results of the paper under review may now be stated as follows. Theorem 1. Every generalized surface L_0 with g-boundary $\lambda \neq 0$ in Λ_0 can be represented in the form $L_0=L'_0+L_0''$, where L_0' is a mixture of $E(\lambda)$ with mass 1 and L_0'' is closed. Theorem 2. Every closed generalized surface L_0 can be represented in the form $L_0=L_0'+L_0''$, where L_0' is a mixture of I and L_0'' is singular. *T. Radó* (Columbus, Ohio).

Gloden, Raoul-François. Sur les systèmes triples orthogonaux. C. R. Acad. Sci. Paris **243** (1956), 1010–1012.

If $z=f(x, y)$ is an analytic surface in Euclidean space of three dimensions, let $p=\partial z/\partial x$ and $q=\partial z/\partial y$. It is known that a triply orthogonal system of surfaces is defined by a system of two partial differential equations of the form: $\phi=pq+Ap+Bq=0$, $\psi=p^2-q^2+\alpha(Gp+Hq)=0$, where A, B, α, G, H, are analytic functions of (x, y, z) such that $\alpha(BG-AH)+1=0$. By the method of Lagrange and Charpit, the compatibility condition for this system is an equation of the form $\Gamma=0$, where Γ is quadratic in p and q with coefficients containing first order partial derivatives of A, B, G, H. Upon applying the usual devices of elimination theory, the author obtains a system of three partial differential equations of the first order in A, B, and Λ, where $G=(\Lambda-1)/4B$, and $H=(\Lambda+1)/4A$. Therefore by a certain existence theorem in partial differential equations, the functions A, B, G, H are determined up to three arbitrary analytic functions of (x, y) only. For any such determinations of A, B, G, H, the system $\phi=0$, $\psi=0$ is completely soluble. Thus all triply orthogonal systems of analytic surfaces may be found in this way. Another method of solving this problem may be found in G. Darboux, Leçons sur les systèmes orthogononaux et les coordonnés curvilignes [2ième éd. Gauthier-Villars, Paris, 1910, p. 4]. *J. De Cicco.*

★ **Норден, А. П. [Norden, A. P.] Теория поверхностей. [Theory of surfaces.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 260 pp. 5.35 rubles.

A rudimentary knowledge of differential geometry is presupposed. The book covers the usual topics by tensor methods (exterior differentiation is not mentioned) and is distinguished by its exactness and discussion of special classes of surfaces although minimal surfaces are treated in less detail than usual. There are 15 chapters: I Curve theory, II Tensor algebra, III A surface and its tangent plane (including envelopes and developable surfaces), IV First fundamental form, V Second fundamental form, VI Surfaces of revolution and their generalizations (surface formed by the orthogonal trajectories of an arc-parameter family of planes and surfaces), VII Ruled surfaces and line congruences, VIII Vector and tensor fields on a surface (includes Laplace fields, parallel displacement, absolute and covariant differentiation), IX Geodesic curvature and geodesics (Liouville and Weingarten surfaces are treated among others), X Nets on a surface (in particular Tchebyshev nets), XI Mapping of surfaces (conformal, geodesic, by inversion, ···; it is notable that the spherical mapping appears here for the first time), XII The Gauss curvature as intrinsic invariant, XIII Surfaces with constant curvature (Theorems in the large, but Hilbert's are not treated in this book), XIV Minimal surfaces, XV Triply orthogonal systems.

H. Busemann (Los Angeles, Calif.).

Svoboda, Karel. Sur une classe de surfaces à l'indicatrice de courbure normale localement sphérique dans un espace à cinq dimensions. Acta Acad. Sci. Čechoslovenicae Basis Brunensis **27** (1955), 373–392. (Czech and Russian summaries)

Let S_5 be a 5-dimensional space of constant curvature c and S a surface in it. The indicatrix of normal curvature of S at a point M of it is defined as the locus of the endpoint of the normal curvature vector of S at M. The author shows that a family of surfaces S (depending on two arbitrary functions) always exists such that the indicatrix of normal curvature of S at each of its points M is a circle formed by the intersection of a sphere whose center is M and a plane whose distance v from M is a constant over S. Indeed, it is proved that surfaces exist with the above property for which the radius r of the circle referred to above remains constant as M varies over S. In this latter case, the relationship $3r^2-v^2-c=0$ is satisfied. The geometry of these surfaces is studied further using auxiliary Riemannian and projected spaces. The discussion divides into two cases according as v^2+c does or does not vanish. The paper employs the method of Cartan's repère mobile. *A. Fialkow.*

Nasu, Yasuo. On angular measure in a metric space. Tôhoku Math. J. (2) **8** (1956), 1–12.

Using the reviewer's approach to angle the paper proves the Gauss-Bonnet theorem for surfaces of constant curvature in "Busemann's sense" (i.e. where, locally,

the loci equidistant from two points are geodesics), evidently under the delusion that these surfaces are more general than surfaces of constant curvature in the usual sense. A proof that these concepts coincide was sketched by the reviewer previously; all details are found in Sections 46, 47, of his "Geometry of geodesics", [Academic Press, New York, 1955; MR 17, 779]. *H. Busemann.*

Backes, F. Sur les surfaces réglées. Mathesis **65** (1956), 177–185.

Détermination des surfaces réglées admettant une ligne de striction donnée. L'étude de leur paramètre de distribution conduit aux courbes de Bertrand où interviennent deux surfaces à paramètre constant. Si trois droites engendrent des surfaces réglées ayant même ligne de striction et même paramètre de distribution, ces droites forment un trièdre de forme invariable. *O. Bottema.*

Helfenstein, Heinz; and Wyman, Max. Geodesic mapping of minimal surfaces. Math. Ann. **132** (1956), 310–327.

A minimal surface admits geodesic mappings on some other surface (not necessarily a minimal surface) which are not isometries or similarities, if and only if it is 1) a plane or a Poisson surface (surface obtained by translating an isotropic straight line along an isotropic curve), 2) similar to one of these types of Liouville surfaces, of which the equations are given, 3) similar to an associate of the catenoid or Lie's imaginary cubic minimal surface. The algebraic and the real surfaces among these minimal surfaces are also indicated. *D. J. Struik.*

Santaló, L. A. Curves on a surface which are extremals of a function of the curvature and the torsion. Abh. Math. Sem. Univ. Hamburg **20** (1956), 216–222. (Spanish)

Given a function $f(\kappa, \tau)$ of the curvature and torsion of a curve, C, lying on a surface, this paper seeks to find those curves, C, for which the variation of $\int_C f(\kappa, \tau)ds$ is zero.

The general result considers families of curves with fixed end-points and common tangents and osculating planes at the end points. It states that this variation will be zero if the angle between the principal normal to the curve and the normal to the surface satisfies an equation involving f, κ, τ and their derivatives. The differential equation of such curves is in general of the sixth order. In special cases this order is reduced, and if $f=c\tau$, the order is two.

When $f=\tau$, the extremal curves coincide with the D-curves; i.e. those for which the osculating sphere is tangent to the surface at each point. If $f=\kappa$, all plane curves are extremals under the stated end-point conditions. Special results are given for closed, non-plane curves and for curves on a sphere. *C. B. Allendoerfer.*

Grincevičyus, K. I. On the hypercomplex of straight lines in the projective space P_4. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 785–788. (Russian)

Dans ce travail on établit le repère canonique mobile $A_1, \cdots, A_5$ de la droite $l=A_1A_2$ du hypercomplexe des droites dans l'espace projectif à quatre dimensions et on construit les objets géométriques fondamentaux et on trouve les relations entre eux.

Soit $\{A_i\}$ le repère mentioné de la droite $l=A_1A_2$, x^i $(i=1, \cdots, 5)$ les coordonnées homogènes du point X par rapport au repère et $m_{\alpha,\beta}$, $n_{\alpha,\beta,\gamma}$, $r_{\alpha,\beta,\gamma,\varepsilon}$ $(\alpha, \beta, \gamma, \varepsilon=3, 4)$ les tenseurs, dont les composantes se trouvent dans les équations obtenues à l'aide de lemma de Cartan en partant des équations qui déterminent le repère $\{A_1\}$.

On obtient les objets fondamentaux:

1) le plan tangent (lA_5), 2) l'hyperplan $x^3-tx^4=0$ qui correspond au point A_1+A_2t, 3) les points invariants (les centres) $x^1A_1+x^2A_2$, $f(x^2, x^1)=6m_{\alpha\beta}x^{5-\alpha}x^{5-\beta}=0$, 4) les hyperplans $f(x^3, x^4)=0$ qui correspondent aux centres de la droite, 5) l'hypersurfaces du 3ième degré

$$f(x^3, x^4)x^5+2n_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma=0,$$

6) l'hypersurface du 4ième degré

$$f(x^3, x^4)(x^5)^{(2)}+4n_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma+r_{\alpha\beta\gamma\varepsilon}x^\alpha x^\beta x^\gamma x^\varepsilon=0.$$

F. Vyčichlo (Praha).

Kovancov, N. I. Canonical tetrahedron of a complex of straight lines in projective space. Ukrain. Mat. Ž. **8** (1956), 140–158. (Russian)

Dans ce mémoire on montre comment on peut simplifier le système des équations différentielles de Pfaff qui définissent l'environ de la droite d'un complex.

En partant des équations

$$dA_i=\omega_{ik}A_k,\ D\omega_{ik}=[\omega_{ij}\omega_{jk}]\quad (i=1, 2, 3, 4)$$

pour le mouvement infinitésimal du tétraèdre $A_1A_2A_3A_4$, conjugé à une droite arbitraire du complexe, on obtient le système

$$\omega_{12}-\omega_{34}=x_1\omega_{23}+x_2\omega_{14}+x_3\omega_{24}$$

$$\omega_{21}-\omega_{43}=x_2\omega_{23}+x_4\omega_{14}+x_5\omega_{24}$$

$$\omega_{22}-\omega_{44}+\omega_{33}-\omega_{11}=x_3\omega_{23}+x_5\omega_{14}+x_6\omega_{24}.$$

Le résultat principal du travail est l'interprétation géométrique des coëfficients $x_1, \cdots, x_6$ dans ce système.

F. Vyčichlo (Praha).

Havlíček, Karel. Eine Bemerkung zur Liniengeometrie der abwickelbaren Flächen. Časopis Pěst. Mat. **81** (1956), 26–37. (Czech. Russian and German summaries)

Soit $p_{ik}(t)$ les coordonnés plückeriennes des droites d'une surface réglée S. La surface S est une développable quand et seulement quand les dérivées dp_{ik}/dt sont les coordonnées de la droite. La surface $dp_{ik}(t)/dt$ est la surface dérivée de la surface p_{ik} primitive.

On voit qu'il existe une infinité des surfaces dérivées de la surface S et une infinité des surfaces primitives de la surface donnée.

L'auteur établit les théorèmes: 1) La courbe de rebroussement de la surface développable est l'asymptotique de la surface dérivée. 2) La surface des tangentes d'une courbe asymptotique de la surface réglée S est la surface primitive de la surface S.

On voit que les droites des surfaces primitives (dérivées) de la surface donnée appartiennent à une congruence parabolique des droites. *F. Vyčichlo* (Praha).

Sancho de San Roman, J. On a new concept of affine breadth of an oval. Rev. Mat. Hisp.-Amer. (4) **16** (1956), 151–171. (Spanish. English summary)

The triangular width of an oval C in E^2 at the point $p \in C$ is the area of the triangle with maximal area inscribed to C and with p as arc vertex. If this area is independent of p then C is said to have constant triangular width. There are other ovals than the ellipses which have constant triangular width; the ellipses are characterized by the fact that the centers of gravity of the maximal triangles coincide. An oval has constant triangu-

lar width if, and only if, the tangent at p is always parallel to the side opposite to p in a maximal triangle with vertex p. *H. Busemann* (Los Angeles, Calif.).

Norden, A. P. The theory of surfaces of a biaffine space. Kazan. Gos. Univ. Uč. Zap. **114**, no. 2 (1954), 13–38. (Russian)

Biaffine spaces B_4 have been introduced by the author in same Zap. **112** (1952). If $\mathbf{a}_i$ $(i=1, 2, 3, 4)$ are coordinate vectors in such space, then a point x^i can be mapped on the points of two complex affine planes, $R: \mathbf{x}-i\tilde{\mathbf{x}}=X\mathbf{A}_1+Y\mathbf{A}_2=(x^1+ix^2)(\mathbf{a}_1-i\mathbf{a}_2)+(x^3+ix^4)(\mathbf{a}_3-i\mathbf{a}_4)$, and $\mathbf{R}: \mathbf{x}+i\tilde{\mathbf{x}}=\bar{X}\mathbf{A}_1+\bar{Y}\mathbf{A}_2$. A plane $\xi=\varepsilon^{ij}[\mathbf{xy}]$ is called analytic if $\xi\tilde{\xi}=\varepsilon^{ij}\varepsilon^{pq}[\mathbf{xy}\tilde{\mathbf{x}}\tilde{\mathbf{y}}]=0$ $(i, j, p, q=1, 2)$. A surface of B_4 $\mathbf{x}=x^i(u^1, u^2)\mathbf{a}_i$ $(i=1, 2, 3, 4)$ can be written in the form $\phi_1(X, \bar{X}, Y, \bar{Y})=0, \phi_2(X\bar{X}Y\bar{Y})=0$. Writing $\varphi=\frac{1}{2}(Y+\bar{Y})$, $\psi=(\frac{1}{2}i)(Y-\bar{Y})$, then if $\partial\varphi/\partial x'=\partial\psi/\partial x^2$, $\partial\varphi/\partial x^2=-\partial\psi/\partial x^1$ the surface is called analytic, since its tangent planes are analytic. Their study thus is reduced to that of curves in the complex plane. — The present paper continues with non-analytic surfaces and their differential geometry. For such surfaces the line at infinity of the tangent plane given by the bivector $\xi=\frac{1}{2}\varepsilon^{ij}[\mathbf{x}_i\mathbf{x}_j]$, $i, j=1, 2$, $\mathbf{x}_i=\partial_i\mathbf{x}$, is skew to its conjugate line $\tilde{\xi}$: $\xi\tilde{\xi}\neq 0$. Thus a normalisation (rigging) can be defined at a point P of the surface by connecting it with $\tilde{\xi}$, thus forming a plane called first normal; the line ξ is called the second normal. If now $\tilde{\mathbf{x}}_1$ $\tilde{\mathbf{x}}_2$ are taken in the first normal, then if we define $\partial_j\mathbf{x}_i=G_{ij}{}^k\mathbf{x}_k+b_{ij}{}^k\tilde{\mathbf{x}}_k$, a covariant derivative is found in $\nabla_j\mathbf{x}_i=b_{ij}{}^k\tilde{\mathbf{x}}_k$, $\nabla_j\tilde{\mathbf{x}}_i=-b_{ij}{}^k\mathbf{x}_k$. Then $\nabla_{[k}b^r_{j]i}=0$, and for the curvature tensor we find $R_{kji}{}^s=2b^r_{i[j}b^s_{k]r}$. If $R_{ji}=R_{kji}{}^k$, then $b^{pq}{}_jR_{pq}=0$, raising and lowering of indices is with the aid of ε_{ik}, ε^{jk}; $\varepsilon_{ik}\varepsilon^{kl}=\delta_i{}^l$. Parallel displacement of a vector on a line η is such that the improper point of η is displaced orthogonal to the second normal on which it is situated: $\eta\xi=\eta\tilde{\xi}=0$. Among the properties studied are geodesic lines, which have their osculating planes perpendicular to the tangent plane. Pseudoasymptotic lines are defined by

$$b_{krs}du^kdu^rdu^s=0.$$

Several others follow, several depending on the curvature tensor. *D. J. Struik* (Cambridge, Mass.).

Terracini, Alessandro. Osservazioni sulle coppie di elementi curvilinei spaziali. Rev. Un. Mat. Argentina **17** (1955), 293–297 (1956).

A further contribution to the study of differential elements (E_2, E_3) of the ordinary projective space [Rend. Mat. e Appl. (5) **14** (1955), 439–454; MR **17**, 527].

Given two curves C, C' having a common point O and tangent t but different osculating planes γ, γ' and a plane α through t, the condition that the projections $\bar{C}, \bar{C}'$ on α of C, C' respectively from points G, G' have a 2nd order contact at O determines a projectivity in the pencil of planes with t as axis: its self-corresponding planes are α and Halphen's principal plane at O (γ, γ' being corresponding plane). The invariant of this projectivity coincides with E. Bompiani's invariant of the elements E_2, E_2' of C, C' and α.

The contact of $\bar{C}, \bar{C}'$ rises to the 3rd order if OG, OG' are corresponding lines in a quadratic transformation in the bundle of lines with O as center. This transformation is a projectivity (supposing γ, γ' not stationary) for a uniquely determined plane α^*: the cross-ratio of $\alpha^*, \gamma, \gamma'$ and the principal plane coincides with another invariant given by this reviewer for elements E_3. *E. Bompiani.*

Egorov, I. P. Equi-affine spaces of third lacunarity. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 1007–1010. (Russian)

L'espace A_n à n dimensions, equiaffin et avec le groupe g_r d'ordre r des mouvements où $n^2-n-2\leq r\leq n^2-n+1$ est l'espace de la troisième lacunarité.

En partant de ses résultats publiés dans mêmes Dokl. (N.S.) **57** (1947), 867–870; **84** (1952), 209–212 [MR **9**, 468; **14**, 318] l'auteur considère les espaces equiaffins A_n avec les groupes d'ordre r pour lesquels

$$n^2-2n+5<r\leq n^2+1.$$

Ces espaces sont nécessairement projectif-euclidiens avec le tenseur de Ricci R_{ij} du rang $h=2$ ou 1.

Dans le cas $h=2$ on emploie la relation connue dans les coordonnés cartesiennes sous la forme

$$\Gamma^i{}_{jk}=-\delta_j{}^i\psi_k-\delta_k{}^i\psi_j \quad (\psi_k=\partial_k\psi)$$

et on cherche la solution du système des équations

$$\text{(a)} \qquad \partial_j\xi_r{}^i=\xi_r{}^i\psi_j+\delta_j{}^i\xi_r{}^k\psi_k \qquad \left(\partial_j=\frac{\partial}{\partial x^j}\right)$$

qui définit le champ des vecteurs contravariants absolument parallèles en environ du point régulier.

On démontre le théorème: La condition nécessaire et sufficante pour l'existence du champ des vecteurs $\xi_\alpha{}^i$ $(\alpha=1, \cdots, s)$ absolument parallèles en environ K du point donné est: Il existe un système des coordonnées cartésiennes x^i dans lequel la fonction ψ a la forme $\psi(x^{s+1}, \cdots, x^n)$.

Dans le cas $h=1$ on a $R_{ij}=\varepsilon\lambda_i\lambda_j$, $\varepsilon=\pm 1$, et λ_i est un vecteur covariant pour lequel on obtient la dérivée covariante $\lambda_{j,i}=\lambda_jp_i+2\lambda_ip_j$. Ici p_j est le vecteur covariant avec la dérivée covariante

$$p_{i,k}=2p_ip_k+\lambda_iT_k+2\lambda_kT_i,$$

où T_j est un vecteur covariant.

Les conditions d'integrabilité du système (a) conduisent aux équations

$$\lambda_i\xi^i=0,\ p_i\xi^i=0,\ T_i\xi^i=0$$

qui montrent que le vecteur ψ_i est le gradient de la fonction $\psi(\alpha^{n-1}, x^n)$ et qu'il existe exactement $(n-2)$ champs des vecteurs ξ^i absolument parallèles.

Les résultats du travail:

1. Il n'existe aucun espace equiaffin A_n avec le groupe complet g_r des mouvements dont l'ordre r satisfait aux inégalités $n^2-2n+5<r<n^2-n-2$.

2. L'espace equiaffin A_n contient le groupe complet g_r des mouvement dont l'ordre r satisfait aux relations $n^2-r-2\leq r\leq n^2-n+1$ quand et seulement quand l'éspace A_n possède $n-2$ champs des vecteurs contravariants absolument parallelès.

Le travail contient aussi certaines généralisations aux cas quand le tenseur R_{ij} a le rang $n-k$ et quand A_n est l'éspace equiprojectif. *F. Vyčichlo* (Praha).

See also: Kuranishi, p. 474; Matschinski, p. 502; Derry, p. 503; Minagawa, p. 503; Leichtweiss, p. 507; Mewes, p. 523; Štepa, p. 535.

Riemannian Geometry, Connections

de Mira Fernandes, A. Di alcune proprietà dei pseudo-estensori. Univ. Lisboa. Revista Fac. Ci. A. (2) **3** (1954–1955), 317–326.

In a recent paper [same Revista **2** (1953), 361–380;

MR 15, 900] the author has defined the pseudo-extensors. Here the most important relations between co- and contravariant pseudo-extensors and extensors are deduced. *J. A. Schouten* (Epe).

Hsiung, Chuan-Chih. Some integral formulas for closed hypersurfaces in Riemannian space. Pacific J. Math. 6 (1956), 291–299.

The purpose of the paper is the generalization to Riemannian spaces of the results previously obtained by the author for hypersurfaces imbedded in a Euclidean space [Math. Scand. 2 (1954), 286–294; MR 16, 849]. Let V^n be a hypersurface imbedded in a Riemannian space R^{n+1} of $n+1$ ($n\geq 2$) dimensions, M_i be the ith mean curvature of V^n at the point P, dA the area element of V^n at P and $p(P)$ be the scalar product of the unit normal vector of V^n at P and the position vector of P with respect to any orthogonal frame in the space R^{n+1} (note: $p(P)$ is defined analytically and is not invariant under a transformation of the local coordinates of V^n at P). Then, the main theorems, slightly modified by the author in a letter to the reviewer, read: 1. Let V^n be a closed orientable hypersurface with $M_1\neq 0$ twice differentiably imbedded in a contractible region of R^{n+1}, then $A+\int_{V^n} M_1 p\, dA=0$; 2. Let V^n be a closed orientable hypersurface of class C^3 imbedded in a contractible region of an R^{n+1} of constant curvature K, then

$$\int_{V^n} M_{n-1}dA+\int_{V^n} M_n p\, dA=0.$$

L. A. Santaló (Buenos Aires).

Petrov, P. I. On differential invariants of conformally flat three-dimensional Riemannian spaces. Kazan. Gos. Univ. Uč. Zap. **114**, no. 2 (1954), 109–122. (Russian)

When the curvature tensor of a Riemannian V_3 is written $B_{\nu\mu\lambda k}=4g_{[\nu[\lambda}\mathfrak{C}_{\mu]k]}$, and $T_{ikj}=2[\mathfrak{C}_{(ik),j}-\mathfrak{C}_{j(i,k)}]$, then $T_{ik}=T_{ki}=T\omega_\mu\varepsilon_k{}^{\lambda\mu}(T_{\omega^\sigma}=0)$ is equivalent to T_{ikj}. The tensors T_{ik}, $C_{ikj}=3L_{(ik,j)}$ form a complete set of orthogonal linear covariants of $\mathfrak{C}_{ik,j}$. When $T_{ik}=0$, the V_3 is conformally euclidean. For such a V_3 a basis of 13 invariants is derived. *D. J. Struik.*

★ **Hachtroudi, Mohsen. Sur les espaces de Riemann, de Weyl et de Schouten.** Imprimerie de l'Université de Teheran, 1956. iv+127 pp.

This book concerns itself with a number of distinct geometric questions in n-dimensional Riemann space and other spaces with a linear connection. The author first deals with the existence of certain fields of multivectors (called "fundamental" by the author) in a Riemann space V_n having metric tensor g_{ij}. It is shown that the integrability conditions for the equations $\xi_{i,j}=g_{ij}\xi$ (where the semicolon designates covariant differentiation with respect to the Christoffel symbols) defining the "fundamental vector" ξ_i are satisfied if V_n ($n>2$) is a space of constant curvature. In this case, the solution for the potential U is $U=(a_ix^i+a)/(K\sum(x^i)^2+1)+A$ and ξ_i is the gradient $\partial U/\partial x^i$. A separate discussion is given for $n=2$. The metric of V_n and the solution for U are obtained also for the special cases where some components of ξ_i vanish. The author also obtains some solutions for the V_n's admitting "fundamental multivector fields" of higher order and finds the expressions for these tensors.

As a generalization of the above equations, the author considers equations of the form $\xi_{i,j}+v_i\xi_j+v_j\xi_i=\lambda g_{ij}$ and $\xi_{i,j}=\lambda R_{ij}+\mu g_{ij}$ (here R_{ij} is the Ricci tensor of V_n). In each case, it is found that any conformally euclidean space V_n ($n\geq 3$) admits fields of vectors ξ_i. The case $n=2$ is treated separately.

Another topic concerns various geometric properties of spaces S_n having a Riemann metric tensor g_{ij} and also an affine connection $\Gamma^h{}_{ij}$. Let $\Gamma^h{}_{ij}=\{{}_i{}^h{}_j\}+\frac{1}{2}g^{hk}(T_{ijk}+T_{jik})+\frac{1}{2}\Gamma^h{}_{ij}$ where the tensor $T^h{}_{ij}$ is the torsion of the space. When the torsion tensor has the form $T^i{}_{kh}=\delta_h{}^iV_k-\delta_k{}^iV_h$ and the Riemann curvature tensor formed from the $\Gamma^h{}_{ij}$ enjoys all the usual symmetry properties, the space is termed "pseudo-Riemannian". Relationships among the various curvature tensors relative to $\{{}_i{}^h{}_j\}$ and $\Gamma^h{}_{ij}$ are derived. The geodesics (relative to g_{ij}) and the paths (relative to $\Gamma^h{}_{ij}$) and other curves are studied. The fundamental forms for a hypersurface are developed. Further topics which are developed in the book concern the geometry of a class of Weyl spaces, the reducibility of certain harmonic functions and others. *A. Fialkow.*

Leichtweiss, Kurt. Das Problem von Cauchy in der mehrdimensionalen Differentialgeometrie. II. Existenz und Eindeutigkeit spezieller Mannigfaltigkeiten. Math. Ann. **132** (1956), 1–16.

This is a sequel to the author's paper on Cauchy's problem [Math. Ann. **130** (1956), 442–474; MR **17**, 1129]. Here the author obtains a theorem which contains as a special case a solution to a generalisation of Bjorling's problem on the existence of minimal surfaces. As a corollary it follows that through an arbitrary, analytic $(m-1)$-dimensional strip of a Riemannian space R_n ($n>m$) there passes exactly one analytic m-dimensional minimal subspace. The remainder of the paper gives results on the existence and characterisation of special manifolds which are natural generalisations of ruled surfaces. As in part I the results are local in character.

T. J. Willmore (Liverpool).

Hano, Jun-ichi; and Ozeki, Hideki. On the holonomy groups of linear connections. Nagoya Math. J. **10** (1956), 97–100.

It is shown that any connected subgroup of the full linear group in n variables is the homogeneous holonomy group of an affine connection on some n-dimensional manifold. In particular, the restricted homogeneous holonomy group need not be closed; an example is given to show it may be non-closed even when the torsion is 0.

W. Ambrose (Cambridge, Mass.).

de Mira Fernandes, A. Direzioni isocliniche nei trasporti lineari. Univ. Lisboa, Revista Fac. Ci. A. (2) **3** (1954–1955), 243–254.

In a V_n one linear connexion for the covariant vectors and one for the contravariant vectors are defined by means of five tensors of valence 3. Along a curve curvatures are defined. Some special cases are discussed.

J. A. Schouten (Epe).

Auslander, Louis. On holonomy covering spaces. Proc. Amer. Math. Soc. **7** (1956), 685–689.

It is shown that there exist, for the n-dimensional torus T^n, n flat affine connections $A_1, \cdots, A_n$, such that the holonomy covering space of A_i [in the sense of L. Auslander and L. Markus, Ann. of Math. (2) **62** (1955), 139–151; MR **17**, 298] is $T^1\times R^{n-1}$. *W. Ambrose.*

Lichnerowicz, André. Espaces homogènes riemanniens et réductibilité. C. R. Acad. Sci. Paris **242** (1956), 1410–1413.

It is proved for a homogeneous Riemannian manifold $V=G/H$, if G is effective and simply connected and H is compact and connected, that G has an invariant subgroup which is a product of groups operating transitively and effectively on the factors of the de Rham decomposition of V. In particular, if G is semi-simple then V has no Euclidean part. *W. Ambrose* (Cambridge, Mass.).

Ku, C. H.; and Su, Buchin. The first and second variations of the volume integral in a space with a multiple areal metric. J. Chinese Math. Soc. (N.S.) **2** (1953), 231–245. (Chinese. English summary)

Let S_K be a space with given K-dimensional areal metric and affine connection, which depends on the position and a supporting K-ple element. If the affine connection depends only on the position, then S_N becomes an ordinary affinely connected space. In this paper the first variation of the "volume" integral of a closed domain of a differentiable K-dimensional variety V_K in S_N, as well as the second variation of the "volume" integral of an extremal variety V_K in S_N, is obtained. *C. C. Hsiung.*

Su, Buchin. Volumentary geometry of an affinely connected space with areal metric. J. Chinese Math. Soc. (N.S.) **2** (1953), 246–257. (Chinese. English summary)

As a continuation of the paper reviewed above, by replacing the affine connection by a volumentary connection derives the second variation of the "volume" integral of an extremal variety V_K in the space S_N.

C. C. Hsiung (Bethlehem, Pa.).

Takeda, Kusuo. On the imbedding of a projective space of connection. Yokohama Math. J. **2** (1954), 95–105 (1955).

It is proved that a necessary and sufficient condition for a curve C in a space of projective connection and the dominant image of C in the corresponding space of dominant connection to admit a deformation of the third order is that the latter space is torsion free. Additional conditions are obtained for admitting a deformation of the fourth order. *C. C. Hsiung* (Bethlehem, Pa.).

Širokov, A. P. Correction to the article "Projectively-Euclidean symmetric spaces". Trudy Sem. Vektor. Tenzor. Anal. **10** (1956), 309–310. (Russian)

Given $S_{\alpha\beta\gamma}+S_{\beta\gamma\alpha}+2S_{\gamma\beta\alpha}=0$, where $S_{\alpha\beta\gamma}=\omega_{\alpha\alpha}\omega_{[\beta\gamma]}$. Then $\omega_{[\alpha\beta]}=0$. This theorem of P. A. Širokov is here provided with a better proof than was given in Trudy Sem. Vektor. Tenzor. Anal. **8** (1950), 73–81, p. 75 [MR **12**, 441]. *D. J. Struik* (Cambridge, Mass.).

See also: Nomizu, p. 489; Laugwitz and Lorch, p. 495; Moisil, p. 498; Kobayashi, p. 503; Svoboda, p. 504.

Complex Manifolds

★ **Baldassarri, M. Algebraic varieties.** Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Heft 12. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. ix+195 pp. DM 36.

The general study of the geometry upon an algebraic variety begins with a fundamental paper by Severi [Rend. Circ. Mat. Palermo **28** (1909), 33–87; dedicated to the algebro-geometric point of view] three years after an important work of transcendental character by Castelnuovo and Enriques [Ann. Sci. Ecole Norm. Sup. (3) **23** (1906), 339–366]. Then, the subject has been standing for many years, excepting for some topological contributions by Lefschetz. But next, during the last twenty-five years, it has flourished magnificently, with a deep study of foundations and the rejection of restrictions on the groundfield, with the introduction of several types of equivalences and canonical varieties, and finally with the utilization of the recent theories on valuation, harmonic integrals, fibre spaces and sheaves (or stacks).

Thus, the birational study of algebraic varieties can nowadays be considered as accomplished in its principal lines, and has given rise to a number of many-sided methods and results, among which, however, a clear coordination was not always to be found. There was therefore a widespread need for an ensemble exposition of the whole theory in its present stage. This important and very difficult task has been carried out successfully in this book, which will remain for a long time to come a main source of information and consultation.

The author has skilfully condensed in comparatively few pages an immense amount of knowledge, stressing at the same time the connections among the different parts of the subject. It is impossible to give here a detailed account of the content; but a general idea of it will be suggested by the following titles of the eleven chapters of the book:

I. A survey of the foundations. II. The resolution of singularities. III. Linear systems. IV. The geometric genus. V. The arithmetic genus. VI. Algebraic and rational equivalence. VII. The Abelian varieties from the algebraic viewpoint, and related questions. VIII. Theory and applications of the canonical systems. IX. The algebraic varieties as complex-analytic manifolds. X. The applications of stack theory to algebraic geometry. XI. The superficial irregularity and continuous systems.

Chapters VII and X are a little scanty; but a book by Hirzebruch on the subject of Ch. X has just appeared [Neue topologische Methoden in der algebraischen Geometrie, Springer, Berlin, 1956, MR **18**, 509], and a book by Andreotti with further developments related to Ch. VII will be published in the same Collection. It may be noticed, moreover, that in the present book the special algebraic, transcendental and topological techniques required are taken for granted, and the exposition is restricted to fundamentals, without omitting, however, to sketch the proofs of many results. All this, while imposed by space restrictions, sometimes makes reading rather difficult. Other causes of difficulty are due to a number of oversights and misprints, to the fact that only some of the definitions, theorems and notations are recalled in the Index, and to the frequent use of abbreviations, not always clear from the context [cf. e.g. form. (2) on p. 163], and sometimes employed in different places with different significance [thus, for instance, the abbreviation "def" is used with four different meanings on pp. 4, 32, 74 and 164]. Here we list a few misprints, where the preceding number and index indicate the page and the line, and the expressions in curled brackets give the corresponding corrections.

48_{-11}: $\varphi(W, m)$ $\{\varphi(W, 0)\}$; 77_{15}: $G_l^r(V)$ $\{G_l^r(U)\}$; 79_{18}: $n-r$-cycles $\{n-r$ cycles$\}$; 92_{15}: $G_1' \subset G_2'$ $\{G_1' \supset G_2'\}$; 128_{-6}: $\int_{C^r} d\varphi$ $\{\int_{C^{r+1}} d\varphi\}$; 129_4: $(n-p)$-forms $\{(2n-p)$-forms$\}$;

137_{-3}: dim $V-1$ {dim $V>1$}; 138_6: Hindernis {hindernis}; 142_7: homomorphism onto {homeomorphism into}; 157_{23}: U of V {U on V}; 157_{24}: $f(z)\circ s(z)$ {$f(z)\cdot s(z)$}; 158_3: $d_j \in H^{2j}(V, Z)$ {$d_j \in H^{2j}(V, Z)$}; 162_{-18} metric {matrix}; 172_{26}: forms {1-forms}; 177_{24}: least {greatest}.

The book contains an ample Bibliography of 54 treatises, monographs and reports and of 372 papers, and it frequently gives many useful bibliographical and historical indications. The latter are usually very accurate, and only in a few cases require revision {thus, e.g., the formula: $24(p_a(V^3)-1)=KC$ on p. 159 is there ascribed to Todd [9], while it appears already in B. Segre [2], form. (60) and (82)}. *Beniamino Segre* (Rome).

Chow, Wei-Liang. On equivalence classes of cycles in an algebraic variety. Ann. of Math. (2) **64** (1956), 450–479.

In this paper the author develops a systematic theory of rational equivalence for algebraic cycles on an algebraic variety V (embedded in projective space). If V is not complete several different theories are possible (absolute, relative, restricted). For simplicity suppose V is complete and non-singular. Let K be a field of definition for V and let X be a cycle on V. We may write $X=X'-X''$ where X' and X'' are positive cycles without common components. The cycle $X_1=X_1'-X_1''$ is a specialization of X over K if there exists a positive cycle Z such that $X_1'+Z$, $X_1''+Z$ are specializations of X', X'' over K (the specialization of positive cycles being defined in terms of the Chow coordinates). If X_1 is rational over K, X rational over a purely transcendental extension $K(u)$ of K, and if X_1 and the zero cycle are both specializations of X over K, we say that X_1 is rationally equivalent to zero. The cycles of dimension s which are rationally equivalent to zero form a subgroup $G_r(V, s)$ in the group $G(V, s)$ of all s-cycles in V. Let the quotient group be $H_r(V, s)$ and let $H_r(V)$ be the direct sum of the $H_r(V, s)$ over all s.

The author proves first that rational equivalence is compatible with intersections, in the sense that one can define a ring structure in $H_r(V)$. He then proves that if $\mathcal{F}: U\to V$ is a regular map it induces a ring homomorphism $\mathcal{F}^{-1}$: $H_r(V)\to H_r(U)$ (in analogy with cohomology). Actually $\mathcal{F}$ need not be a genuine map, it is sufficient if it is a suitable cycle on $U\times V$.

The modifications necessary when V is not complete or has singularities are fully dealt with. The main lemma of the paper rests on a geometric construction in the ambient projective space. For this reason the theory does not cover abstract varieties in the sense of Weil.

M. Atiyah (Cambridge, Eng.).

See also: Hirzebruch, p. 509; Serre, p. 511.

Algebraic Geometry

★**Hirzebruch, F. Neue topologische Methoden in der algebraischen Geometrie.** Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 9. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. viii+165 pp. DM 30.80.

This monograph is an exposition of the main ideas and concepts centering around the fundamental Riemann-Roch theorem for algebraic manifolds of arbitrary dimension, a theorem whose proof has recently been found by the author [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 110–114; MR **17**, 535] and which is presented here in complete form for the first time. An algebraic manifold is understood to mean a compact complex manifold which can be imbedded complex-analytically without singularities in a complex projective space of suitably chosen dimension.

The first chapter is an extremely clear, well-written exposition of those aspects of the theory of characteristic classes and sheaves which are required in the sequel. In particular, the author's systematic treatment of characteristic classes (Pontrjagin and Chern classes) is in itself an important contribution to the general theory. The notation and formalism of multiplicative sequences of polynomials, introduced at the beginning of the first chapter, form the basis of the author's treatment and may be described as follows. Let B denote a commutative ring with unit 1, let $p_1, p_2, \cdots$ be indeterminates and set $p_0=1$. The ring $\mathfrak{B}=B[p_1, p_2, \cdots]$ of polynomials in the p_i with coefficients in B is graded by assigning to p_j the weight j: $\mathfrak{B}=\sum \mathfrak{B}_k$, where $\mathfrak{B}_k$ is the module of polynomials of weight k. A sequence $\{K_j\}$ of polynomials in the unknowns p_i with $K_0=1$, $K_j \in \mathfrak{B}_j$ ($j=0, 1, 2, \cdots$), is a multiplicative sequence if the formal identity $\sum p_j''z^j=(\sum p_jz^j)\cdot(\sum p_j'z^j)$ where z, p_j, p_j' are indeterminates, implies $\sum K_j(p'')z^j=(\sum K_i(p)z^i)\cdot(\sum K_i(p')z^i)$, where $K_i(p)=K_i(p_1, \cdots, p_i)$ etc. To each formal power series $Q(z)=\sum_{i=0}^{\infty} b_iz^i$, $b_0=1$, $b_i \in B$, there corresponds a multiplicative sequence $\{K_j\}$ of polynomials and conversely. This correspondence is established by splitting (formally)

$$1+p_1z+p_2z^2+\cdots+p_mz^m$$

into a product $\prod_{i=1}^{m}(1+\beta_iz)$ of linear factors and using the product $\prod_{i=1}^{m}Q(\beta_iz)$ to define the K_j for $j\leqq m$. It is also convenient to replace p_i by c_i, z by x, β_i by γ_i (with $c_0=p_0=1$), and to introduce the relations $z=x^2$, $\beta_i=\gamma_i^2$. The most important multiplicative sequences are: (1) $\{L_j\}$ defined by $Q(z)=z^i/\operatorname{tgh} z^i$; (2) $\{A_j\}$ defined by $2z^i/\sinh 2z^i$; (3) $\{T_j(y; c_1, \cdots, c_j)\}$ defined by

$$Q(y; x)=x(y+1)/[1-\exp(-x(y+1)]-yx.$$

The second chapter contains an outline of the deep theory of "cobordisme" of R. Thom which is applied by the author to prove that the index $\tau(M)$ of an oriented compact differentiable manifold M of dimension $4k$ is a linear combination of Pontrjagin numbers, namely: $\tau(M)=L_k(p_1, \cdots, p_k)[M]$, where $\{L_j\}$ is the sequence (1) and $p_i \in H^{4i}(M, \mathbf{Z})$ is the ith Pontrjagin class of M. The proof of the Riemann-Roch theorem is reduced to this basic result which is called the "index theorem".

In the third chapter, properties of the virtual Todd genus are derived and a method for splitting vector bundles is developed (splitting method). The splitting method together with the calculus of virtual genera provide the main tool for the reduction of the Riemann-Roch theorem to the index theorem. Let M_n denote an almost-complex manifold of complex dimension n with the natural orientation, and let c_i be the ith Chern class of M_n (ith Chern class of the tangent $U(n)$-bundle of M_n). The generalized Todd genus $T_y(M_n)$ of M_n (shortly T_y-genus) is the polynomial of degree n in y with rational coefficients which is equal to $T_n(y; c_1, \cdots, c_n)[M_n]$, where $\{T_j(y; c_1, \cdots, c_j)\}$ is the multiplicative sequence (3). Let $T_y(M_n)=\sum_{p=0}^{n}T^p(M_n)y^p$. The Todd genus $T(M_n)$ of M_n is the number $T_0(M_n)=T^0(M_n)$, while $T_{-1}(M_n)=\sum_{p=0}^{n}(-1)^pT^p(M_n)=c_n[M^n]$ is the Euler-Poincaré characteristic of M_n, and

$$T_1(M_n)=\sum_{p=0}^{n}T^p(M_n)=\tau(M_n)$$

is the index of M_n (assumed to vanish for odd n) — a fact which is a consequence of the index theorem. If we set $Q(y;x)=x/R(y;x)$ we obtain from (3):

$$R(y;x)=[\exp(x(y+1))-1]/[\exp(x(y+1))+y].$$

In particular $R(1;x)=$ tgh x, $R(-1;x)=x(1+x)^{-1}$, $R(0,x)=1-e^{-x}$. Given $v_1,\cdots,v_r\in H^2(M_n,\mathbf{Z})$, the virtual T_y-genus of the r-tuple $(v_1,\cdots,v_r)$ is defined to be

$$T_y(v_1,\cdots,v_r)_M=\kappa_n[R(y;v_1)\cdots R(y;v_r)\sum_{j=0}^{\infty}T_j(y;c_1(M_n),\cdots)],$$

where κ_n has the following meaning: let $u\in H^*(M_n)$ and let $u^{(n)}$ be the $2n$-dimensional component of u; then $\kappa_n[u]=u^{(n)}[M_n]$. The virtual T_y-genus is a polynomial in y of degree $n-r$ with rational coefficients,

$$T_y(v_1,\cdots,v_r)_M=\sum_{p=0}^{n-r}T^p(v_1,\cdots,v_r)_M y^p,$$

which satisfies the functional equation

$$T_y(v_1,\cdots,v_r,u+v)=T_y(v_1,\cdots,v_r,u)+T_y(v_1,\cdots,v_r,v)$$
$$+(y-1)T_y(v_1,\cdots,v_r,u,v)-yT_y(v_1,\cdots,v_r,u,v,u+v)$$
$$(v_1,\cdots,v_r,u,v\in H^2(M_n,\mathbf{Z})).$$

For $y=1$, 0, -1 we obtain functional equations for the virtual index, the virtual Todd genus, and the virtual Euler-Poincaré characteristic respectively. The T_y-characteristic of a continuous $GL(q,\mathbf{C})$-bundle ξ over M_n is defined as follows. Let $d_0,d_1,\cdots,d_q$ be the Chern classes of ξ, $c_0=1,c_1,\cdots,c_n$ the Chern classes of M_n, consider the formal splittings $\sum_{i=0}^{q}c_ix^i=\prod_{i=1}^{n}(1+\gamma_ix)$, $\sum_{i=1}^{q}d_ix^i=\prod_{i=1}^{q}(1+\delta_ix)$, set

$$t_y(\xi)=\exp((1+y)\delta_1)+\exp((1+y)\delta_2)+\cdots+\exp((1+y)\delta_q$$
$$(t_y(\xi)\in H^*(M_n,\mathbf{Z})\otimes\mathbf{Q}[y]),$$

and define the virtual T_y-characteristic of ξ with respect to the virtual manifold $(v_1,\cdots,v_r)$, $v_i\in H^2(M_n,\mathbf{Z})$ $(i=1,\cdots,r)$ to be

$$T_y(v_1,\cdots,v_r|,\xi)_M=\kappa_n[t_y(\xi)\prod_{i=1}^{r}R(y;v_i)\sum_{j=0}^{\infty}T_j(y;c_1(M_n),\cdots)].$$

If ξ is the trivial $GL(q,\mathbf{C})$-bundle, then

$$T_y(v_1,\cdots,v_r|,\xi)=qT_y(v_1,\cdots,v_r).$$

The T_y-characteristic of ξ satisfies the same functional equation as the T_y-genus.

Now let $\Delta(q,\mathbf{C})$ denote the subgroup of those matrices of $GL(q,\mathbf{C})$ which leave fixed all r-dimensional linear subspaces L_r of $\mathbf{C}_q$ defined by $z_{r+1}=z_{r+2}=\cdots=z_q=0$ ($z_1,z_2,\cdots,z_q$ the coordinates of $\mathbf{C}_q$); $\Delta(q,\mathbf{C})$ is the group of triangular matrices of $GL(q,\mathbf{C})$ (all coefficients beneath the diagonal vanish). Let X be a complex manifold. An element $\xi\in H^1(X,G_\omega)$, where G_ω denotes the sheaf of germs of holomorphic maps of X into the complex Lie group G, will be called a complex- analytic G-bundle, and a fibre bundle belonging to the equivalence class ξ will be said to be associated to ξ. Let ξ be a $GL(q,\mathbf{C})$-bundle over X, L a complex-analytic principal bundle over X which is associated to ξ, $E=L\times_G G/\Delta(q,\mathbf{C})$ the bundle associated to L with the flag manifold

$$F(q)=GL(q,\mathbf{C})/\Delta(q,\mathbf{C})$$

as fibre, and let $\phi\colon E\to X$ be the projection. Then E is a complex manifold, ϕ is an analytic mapping of E onto X, and the structure group of L lifted up to E (bundle induced by ϕ) can be reduced complex-analytically to $\Delta(q,\mathbf{C})$ in a natural way giving rise to q diagonal complex-analytic $\mathbf{C}^*$-bundles which will be denoted by $\xi_1,\xi_2,\cdots,\xi_q$ in their natural order. Denote by e_0 the point in $F(q)$ defined by the coset $\Delta(q,\mathbf{C})$. The group $\Delta(q,\mathbf{C})$ operates by left translation on $GL(q,\mathbf{C})/\Delta(q,\mathbf{C})$ but keeps e_0 fixed; hence $\Delta(q,\mathbf{C})$ operates on the tangent space $\mathbf{C}_m(e_0)$ of $e_0\in F(q)$ and this gives an analytic homomorphism $h\colon\Delta(q,\mathbf{C})\to GL(m,\mathbf{C})$. Let $T(q)$ be thetangential principal fibre bundle of the complex flag manifold $F(q)$; the group $G=GL(q,\mathbf{C})$ operates by left translation on $F(q)$ and therefore on $T(q)$. Hence we have the fibre bundle $L\times_G T(q)$ over X associated to L which, lifted up to E by ϕ, is a bundle over E with fibre $GL(m,\mathbf{C})$ the corresponding $GL(m,\mathbf{C})$ - bundle will be denoted by ξ^Δ. The structure group of ξ^Δ can be reduced complex - analytically to $\Delta(m,\mathbf{C})$ (for example, by using a theorem of Lic since $\Delta(q,\mathbf{C})$ is soluble and the map $\mathbf{h}$ is holomorphic — a remark of A. Borel). This reduction gives rise to m diagonal complex-analytic $\mathbf{C}^*$-bundles equal to $\xi_i\otimes\xi_j^{-1}$ $(i>j)$ with the following ordering: $\xi_i\otimes\xi_j^{-1}$ precedes $\xi_{i'}\otimes\xi_{j'}^{-1}$ if $j>j'$ or $j=j'$ and $i<i'$. Let $\gamma_i=c_1(\xi_i)\in H^2(E,\mathbf{Z})$; then it follows that the Chern class $c(E)$ of E is given by the formula $c(E)=\phi^*c(X)\cdot\prod_{q\geq i>j\geq 1}(1+\gamma_i-\gamma_j)$. In particular, if L is the tangential $GL(n,\mathbf{C})$-bundle of the complex manifold X (of complex dimension n), we set $E=X^\Delta$ and in this case $c(X^\Delta)=\prod_{i=1}^{n}(1+\gamma_i)\cdot\prod_{n\geq i>j\geq 1}(1+\gamma_i-\gamma_j)$. Finally a complex manifold X will be called a complex-analytic split manifold if the structure group $GL(n,\mathbf{C})$ of its tangent bundle can be reduced complex-analytically to $\Delta(n,\mathbf{C})$ (but the tangent bundle is then not in general the complex-analytic Whitney sum of $\mathbf{C}^*$-bundles). The Chern class of a complex-analytic split manifold X is given by $c(X)=\prod_{i=1}^{n}(1+a_i)$, where $a_i=c_1(\xi_i)$, $\xi_1,\cdots,\xi_n$ the $\mathbf{C}^*$-bundles defined by the $\Delta(n,\mathbf{C})$-bundle. For a complex-analytic split manifold X the Todd genus $T(X)$ may be expressed in terms of virtual genera by the formula

$$\text{(I)}\quad(1+y)^nT(X)=\sum_{i=0}^{n}y^i\sum_{1\leq i_1<\cdots<i_i\leq n}T_y(a_{i_1},\cdots,a_{i_i})_X.$$

This formula, whose proof is an almost immediate consequence of the definition of T_y-genus, is used in reducing the Riemann-Roch theorem to the index theorem.

Similar considerations must also be developed for the Euler-Poincaré characteristic $\chi(X,W)$ of the cohomology of X with coefficients in the sheaf of germs of holomorphic sections of a complex-analytic vector bundle W over X. In the fourth and final chapter of the book, a virtual χ_y-characteristic is defined which is the analogue in sheaf cohomology of the T_y-characteristic. Let E be a ring which is an extension of the integers $\mathbf{Z}$ and has the integer 1 as unit, and let $\mathbf{Z}\{y\}$, $E\{y\}$ be the rings of formal power series with coefficients in $\mathbf{Z}$, E respectively. The rings $E\{y\}$ and $\mathbf{Z}\{y\}$ are regarded as $\mathbf{Z}\{y\}$-modules, and a d-homomorphism $h\colon E\{y\}\to\mathbf{Z}\{y\}$ is by definition a $\mathbf{Z}\{y\}$-homomorphism. Given an additive homomorphism $h_0\colon E\to\mathbf{Z}\{y\}$, there is one and only one d-homomorphism $h\colon E\{y\}\to\mathbf{Z}\{y\}$ which coincides with h_0 on E. Moreover given a d-homomorphism $h\colon E\{y\}\to\mathbf{Z}\{y\}$ and a fixed element $t\in E\{y\}$, a d-homomorphism h_t is defined by $h_t(u)=h(tu)$, $u\in E\{y\}$. If, for d-homomorphisms h and h' of $E\{y\}$ into $\mathbf{Z}\{y\}$, there exists an element $t\in E\{y\}$ with $h'(u)=h(tu)$ for all $u\in E$, then $h'=h_t$. Now let $f_1,\cdots,f_r,w$ be unknowns, and consider over $\mathbf{Z}$ the ring $\bar{E}$ generated by these unknowns and by $f_1^{-1},f_2^{-1},\cdots,f_r^{-1}$. A d-

homomorphism of $E\{y\}$ into $\mathbf{Z}\{y\}$ is determined by its values on the products $w^\mu f_1^{\lambda_1}\cdots f_r^{\lambda_r}$, where $\mu, \lambda_1, \cdots, \lambda_r$ are integers, $\mu \geqq 0$. Denote by V a compact complex manifold, by $F_1, \cdots, F_r$ complex-analytic line bundles over V, by W a complex-analytic vector bundle over V, and let 1 denote the trivial vector bundle, $\mathbf{T}^{(p)}$ the complex-analytic vector bundle of tangential covariant p-vectors of X. Define two d-homomorphisms h and $\hat{h}$ of $E\{y\}$ into $\mathbf{Z}\{y\}$ by the formulas:

$$h(w^\mu f_1^{\lambda_1}\cdots f_r^{\lambda_r}) = \chi(V, W^\mu \otimes F_2^{\lambda_2} \otimes \cdots \otimes F_r^{\lambda_r}),\ h(1) = \chi(V);$$

$$\hat{h}(w^\mu f_1^{\lambda_1}\cdots f_r^{\lambda_r}) = \chi_y(V, W^\mu \otimes F_1^{\lambda_1} \otimes F_1^{\lambda_2} \otimes \cdots \otimes F_r^{\lambda_r}),\ \hat{h}(1) = \chi_y(V).$$

Here $\chi_y(V, W) = \sum_{p=0}^{n} \chi^p(V, W) y^p$, $\chi_y(V) = \sum_{p=0}^{n} \chi^p(V) y^p$, where $\chi^p(V, W) = \chi(V, W \otimes \mathbf{T}^{(p)})$, $\chi^p(V) = \chi^p(V, 1)$. The virtual χ_y-characteristic of the vector bundle W restricted to the virtual manifold $(F_1, \cdots, F_r)$ is defined by the formulas

$$\chi_y(F_1, \cdots, F_r, W) = \hat{h}_V\Big(w \prod_{i=1}^{r} \frac{1 - f_i^{-1}}{1 + y f_i^{-1}}\Big).$$

If S is a non-singular divisor of V, $F_1 = \{S\}$ the corresponding line bundle, it is shown that

$$\chi_y(F_1, \cdots, F_r, W)_V = \chi_y((F_2)_S, \cdots, (F_r)_S, W_S)_S.$$

Let

$$\chi_y(F_1, \cdots, F_r, W) = \sum_{p=0}^{\infty} \chi^p(F_1, \cdots, F_r, W)_V y^p,$$

$$\chi(F_1, \cdots, F_r, W)_V = \chi^0(F_1, \cdots, F_r, W)_V = h_V(w \prod_{i=1}^{r} (1 - f_i^{-1})).$$

Finally, suppose that V is a complex-analytic split manifold of complex dimension n with the diagonal complex-analytic line bundles $A_1, \cdots, A_n$, and let W be a complex-analytic vector bundle over V; then

$$\text{(II)}\quad (1+y)^n \chi(V, W) = \sum_{i=0}^{n} y^i \sum_{i_1 < \cdots < i_i} \chi_y(A_{i_1}, \cdots, A_{i_i}, W).$$

We remark that the χ_y-characteristic satisfies a functional equation analogous to that for the T_y-characteristic.

Now let $f_1, \cdots, f_r$ be the cohomology classes of the complex-analytic line bundles $F_1, \cdots F_r$ over V, let the vector bundle W be regarded as a continuous $\mathrm{GL}(q, \mathbf{C})$-bundle ξ, and write $T_y(F_1, \cdots, F_r, W)_V = T_y(f_1, \cdots, f_r, \xi)_V$, $T_y(V, W) = T_y(V, \xi)$. The most general formulation of the Riemann-Roch theorem is as follows: Suppose that V is an algebraic manifold, and let $F_1, \cdots, F_r$ be complex line bundles over V, W a complex-analytic vector bundle over V; then (i) $\chi_y(F_1, \cdots, F_r, W)_V = T_y(F_1, \cdots, F_r, W)_V$. Taking $r=0$, $y=0$, we obtain (ii) $\chi(V, W) = T(V, W)$. We may still further specialize by taking W to be a complex line bundle F, in which case (iii) $\chi(V, F) = T(V, F)$. Finally, taking $F=1$ (trivial line bundle), we obtain (iv) $\chi(V) = T(V)$. There is little real difference in depth between these various statements; it is almost as difficult to prove (iv) as to prove (i). The statement (ii) is the one which is usually called the Riemann-Roch theorem.

The proof, which occupies the last pages of the author's book, follows fairly quickly on the basis of the machinery described above. For example, formula (iv) may be proved in the following steps. (a) Reduction to algebraic split manifolds. (b) By a theorem of Hodge, true for all compact Kähler manifolds, $\chi_1(V) = \tau(V)$ (τ=index); by the induction principle of algebraic geometry, χ_1 and τ are shown to coincide in the virtual case. (c) As already pointed out, $T_1(V) = \tau(V)$, and T_1 and τ coincide in the virtual case also. (d) Take $y=1$, $W=1$ (trivial bundle); then formulas (I) and (II) imply that $\chi(V) = T(V)$ for split manifolds and hence, by (a), in general.

Thus formulas (I) and (II) with $y=1$ are the means of reducing the Riemann-Roch theorem to the index theorem which in turn depends on Thom's theory of "cobordisme".

Two obvious questions arise. First, is it possible to give an algebraic proof of the Riemann-Roch theorem which does not make use of Thom's deep theory? Second, is the Riemann-Roch theorem true for a class of manifolds more general than algebraic, for example compact Kähler? There is evidence that the Riemann-Roch theorem is true more generally, but the fact that an induction principle is lacking for non-algebraic manifolds is a serious obstacle in the way of finding a proof. *D. C. Spencer.*

Serre, Jean-Pierre. Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier, Grenoble **6** (1955–1956), 1–42.

Soit X une variété algébrique définie sur le corps complexe C. On peut étudier X de deus points de vue: 1° algébrique (on s'intéresse aux anneaux locaux de fonctions rationnelles, aux applications rationelles, ou aux applications „régulières" de X); 2° analytique, ou „transcendant" (on s'intéresse aux fonctions holomorphes et aux applications analytiques de X considéré comme espace analytique). Ce mémoire a pour but de comparer les deux points de vue; le classique théorème de Chow, qui dit que tout sous-espace analytique fermé d'une variété algébrique projective est une variété algébrique, est un exemple de l'équivalence essentielle des deux théories. D'autres résultats de ce genre sont établis ici. Voici un résumé incomplet des divers paragraphes.

§ 1: rappel de généralités concernant les „espaces analytiques".

§ 2: toute variété algébrique X (au sens général de A. Weil) définit un espace analytique X^h et une bijection $X \to X^h$ qui est continue quand on munit X de la topologie de Zariski et X^h de la topologie usuelle. Pour que X soit complète, il faut et il suffit que X^h soit compacte. On a un homomorphisme $\mathcal{O}_x \to \mathscr{H}_x$ des anneaux locaux (algébrique, resp. holomorphe) en un point $x \in X$, et l'homomorphisme $\hat{\mathcal{O}}_x \to \hat{\mathscr{H}}_x$ des anneaux complétés est un isomorphisme. Ceci permet d'appliquer au couple d'anneaux $(\mathcal{O}_x, \mathscr{H}_x)$ la théorie des „couples plats" traitée en Annexe, et de comparer "dimension algébrique" et „dimension analytique". Si Y est une sous-variété algébrique (localement fermée) de X, toute fonction holomorphe en $x \in Y$ qui s'annule sur Y appartient à l'idéal de $\mathscr{H}_x$ engendré par les fonctions rationnelles, définies en x, qui s'annulent sur Y. Signalons la prop. 9 (démontrée par Samuel): si une application régulière $f: X \to Y$ induit un isomorphisme $X^h \to Y^h$ des espaces analytiques associés, f est un isomorphisme birégulier.

§ 3: tout faisceau algébrique $\mathscr{F}$ sur X définit un faisceau analytique $\mathscr{F}^h$ sur X^h et une application $\mathscr{F} \to \mathscr{F}^h$; $\mathscr{F}^h$ est un foncteur exact de $\mathscr{F}$, et l'application $\mathscr{F} \to \mathscr{F}^h$ est injective; si $\mathscr{F}$ est algébrique-cohérent, $\mathscr{F}^h$ est analytique-cohérent. Pour tout entier $q \geqq 0$, on a un homomorphisme naturel des groupes de cohomologie ε: $H^q(X, \mathscr{F}) \to H^q(X^h, \mathscr{F}^h)$. Les trois théorèmes fondamentaux concernent le cas où X est une variété algébrique projective (i.e. plongeable dans un espace projectif, avec éventuellement des singularités): (1) si $\mathscr{F}$ est algébrique-cohérent sur X, ε est un isomorphisme; (2) si $\mathscr{F}$ et $\mathscr{G}$ sont algébriques-cohérents sur X, tout homomorphisme analytique

$\mathfrak{F}^h \to \mathfrak{G}^h$ provient d'un homomorphisme algébrique $\mathfrak{F} \to \mathfrak{G}$; (3) tout faisceau analytique-cohérent sur X^h est isomorphe à un $\mathfrak{F}^h$, où $\mathfrak{F}$ est algébrique-cohérent sur X, et $\mathfrak{F}$ est unique (à un isomorphisme algébrique près).

§ 4: utilisant un lemme de Chow, l'auteur prouve que tout sous-ensemble analytique compact d'une variété algébrique (quelconque, non nécessairement projective) est algébrique, et que toute application holomorphe d'une variété algébrique compacte dans une variété algébrique est algébrique-régulière (corollaire: unicité de la structure de variété algébrique donnant naissance à une structure d'espace analytique compact). Soit G un groupe algébrique; soit $\mathfrak{G}$ (resp. $\mathfrak{G}^h$) le faisceau des germes d'applications régulières (resp. holomorphes) de X (resp. X^h) dans G; on a une application ε: $H^1(X, \mathfrak{G}) \to H^1(X^h, \mathfrak{G}^h)$ des classes de fibrés principaux algébriques dans les classes de fibrés principaux analytiques (de groupe G). Si X^h est compacte, ε est injective, et même tout isomorphisme analytique de fibrés algébriques de base X^h est un isomorphisme algébrique. Si X est algébrique projective, ε n'est pas nécessairement une bijection; toutefois ε est une bijection dans les cas suivants: (α) G est le groupe additif C; (β) $G=\mathrm{GL}_n(C)$, ce qui, pour $n=1$, permet d'étendre aux variétés normales une théorème prouvé par Kodaira-Spencer quand X est non-singulière: tout fibré analytique de groupe C^* provient d'un diviseur; (γ) G est un sous-groupe de $\mathrm{GL}_n(C)$ tel qu'il existe une section rationnelle $\mathrm{GL}_n(C)/G \to \mathrm{GL}_n(C)$ (il en est ainsi quand G est résoluble, ou est l'un des groupes $\mathrm{Sp}_n(C)$ et $\mathrm{SL}_n(C)$).

L'Annexe est consacrée aux modules plats et aux couples plats d'anneaux, avec application à la théorie des anneaux locaux. *H. Cartan* (Paris).

Severi, Francesco. Frammenti ricomposti e integrati. I. Le caratteristiche delle coniche nello spazio. Rev. Un. Mat. Argentina **17** (1955), 251–264 (1956).

L'extension de la théorie des caractéristiques aux coniques de l'espace ordinaire présente des difficultés remarquables et ouvre des questions qui ne sont pas encore résolues. Il faut trouver avant tout un modèle projectif convenable du système Σ dont les éléments sont les ∞^8 coniques de l'espace; l'A. pense qu'il ne soit pas possible de trouver un modèle qui soit dépourvu de points singuliers et qui en même temps soit en correspondance birationnelle sans aucune exception avec les éléments de Σ. Le modèle utilisé ici s'obtient en considérant une conique γ quelconque comme l'intersection d'un plan avec un cône du second ordre ayant pour sommet un point 0 fixé arbitrairement dans l'espace; les plans de l'espace et les cônes du second ordre de sommet 0 forment deux systèmes linéaires des dimensions 3 et 5; on voit donc que les coniques γ peuvent être représentées birationnellement sur les points d'une variété de Segre produit d'un S_3 par un S_5. Le modèle ainsi obtenu est non singulier et rationnel, mais la représentation présente des éléments exceptionnels, provenant des coniques dont le plan passe par 0. Un autre modèle est donné, en dualisant, par la variété de tous les cônes du second ordre, mais il est singulier; un autre modèle encore peut être obtenu en considérant le système de tous les complexes quadratiques de droites appuyées sur une conique; tous les deux sont plus compliqués que le premier. En faisant usage de celui-ci, comme on connaît la base des hypersurfaces pures tracées sur une variété de Segre, on trouve sans difficulté que, si l'on n'exclut pas des solutions dégénérées, alors les caractéristiques simples des coniques irréductibles sont deux, c'est-à-dire la condition μ pour que le plan de la conique γ passe par un point donné et la condition ν pour la conique γ rencontre une droite donnée. Dans les mêmes hypothèses, toutes les conditions de dimension $d \leq 7$ s'expriment par des symboles de la forme $\mu^i\nu^j$, où $i, j = 0, 1, \cdots, d$ et $i+j=d$. L'A. considère comme exemple le problème bien connu du nombre des coniques qui rencontrent huit droites données. Le nombre est 92. On le peut calculer en donnant aux droites données des positions particulières convenables; mais il reste à démontrer que les solutions obtenues dans le cas particulier sont des solutions simples. Enfin l'expression de la variété singulière W de la représentation considérée par la base des hypersurfaces de $S_3 \times S_5$; on trouve $W \equiv 2I$, où I est l'hypersurface $S_2 \times S_5$. *E. Togliatti* (Genova).

Rosati, Luigi Antonio. Sul numero dei punti di una superficie cubica in uno spazio lineare finito. Boll. Un. Mat. Ital. (3) **11** (1956), 412–418.

The present paper deals with some cubic surfaces, F, in a projective 3-dimensional space over a Galois field $\gamma = \mathrm{GF}[p^h]$. It determines the number N of points of F in γ in the following four cases:

(i) F contains a straight line r in γ, without being a ruled surface; (ii) F contains a double point in γ; (iii) F is a cubic ruled surface; (iv) F contains two skew lines in a quadratic extension of γ. Thus, for instance, in case (i) it is proved that

$$N = q^2 + 1 + (t - s - g + 2)$$

(whence $q^2 - 5q + 1 \leq N \leq q^2 + 7q + 1$), where $q = p^h$, t and s denote the numbers of planes through r in γ whose residual intersection with F consists of two distinct lines in γ or in a quadratic extension of γ respectively, and g is the number of double points of F in γ lying on r.

{Some of the arguments at the end of no. 2 (concerning extensions of γ in which $N = q^2 + 7q + 1$) are incorrect.} *B. Segre* (Rome).

Snapper, E. Higher-dimensional field theory. II. Linear systems. Compositio Math. **13** (1956), 16–38.

Étant donné un module M de E (cf. ci-dessus) l'ensemble des modules $a^{-1}M$ ($a \in M$, $a \neq 0$) est appelé une classe projective; la donnée d'une telle classe est équivalente à celle d'un modèle projectif d'une sous extension de E; si C est cette classe, la sous extension correspondante est $F(M)$ où M est n'importe quel élément de C; on appelle dimension de C le degré de transcendance de cette extension $F(M)$ (ou $F(C)$) de M. Etant donnée une valuation v de E (triviale sur F) et une classe projective C, il existe $M \in C$ tel que M soit contenu dans l'anneau R_v de v; si l'on note h_v l'homomorphisme canonique de R_v sur son corps quotient E_v, les modules $h_v(M)$ où $M \in C$ et $M \subset R_v$ forment une classe projective de E_v sur F, qu'on note $h_v(C)$ et qu'on appelle le centre de v sur C; si v est de diviseur (i.e. si $\dim(E_v) = \dim(E) - 1$), on dit qu'il est un première espèce sur C si $\dim(h_v(C)) = \dim(C) - 1$. Etant données deux classes projectives C, C', les modules M, M' où $M \in C$ et $M' \in C'$ engendrent une classe projective, notée CC'; les corps $F(C)$ et $F(C^h)$ ($h \geq 1$) sont égaux; si $\dim(C) \geq 1$ et si $h \geq 1$, les classes C et C^h ont mêmes diviseurs de première espèce. Etant donnée une classe projective C, les fermetures intégrales $i(M)$ où $M \in C$ forment une classe projective, notée $i(C)$, qui a même dimension et mêmes diviseurs de première espèce (si $\dim(C) \geq 1$) que C; pour $M \in C$ et $h \geq 1$ le module $i(M^h)$ est l'ensemble des éléments x de E tels que $v(x) \geq \min_{y \in M}(v(y))$ pour toute valuation v d'un ensemble $\mathfrak{V}$ de valuations de E contenant tous les diviseurs de première espèce de C.

Soit maintenant C une classe de dimension maximale ($\dim(C)=\dim(E)$), et $\mathfrak{B}$ l'ensemble de ses diviseurs de première espèce; on appelle cycle toute combinaison linéaire formelle d'éléments de $\mathfrak{B}$, et on définit, à la manière ordinaire, le cycle, le cycle des zéros et le cycle des poles d'un élément $x\neq 0$ de E, et le cycle $G(M)$ d'un module $M\neq(0)$ de E ($G(M)=\sum_v \min_{y\in E}(v(y))\cdot v$). Etant donné un cycle G, l'ensemble $L(G)$ des éléments x de E tels que $G(x)\geqq -G$ est un module; la réunion des $L(hG)$ ($h\geqq 0$) est un sous anneau de E, et le 14-ième problème de Hilbert équivaut à la conjecture qu'il est engendré sur F par un nombre fini d'éléments. Etant donnés un cycle G et un module M tels que $G(M)+G\geqq 0$, l'ensemble des cycles $G(x)+G$ où $x\in M$, $x\neq 0$ (qui sont positifs) s'appelle un système linéaire sur C, et se note $g(M;G)$; on dit que $g(M;G)$ est décent si $L(hG)=i(M^h)$ pour tout $h\geqq 1$; alors $\dim_F(L(hG)$ est un polynôme en h pour h grand, et la réunion des $L(hG)$ est un anneau engendré sur F par un nombre fini d'éléments; si $g(M;G)$ n'a pas de cycle fixe (c'est à dire si $G=-G(M)$), il est décent; il en est de même s'il n'a pas de point base.

P. Samuel.

Snapper, E. Higher-dimensional field theory. III. Normalization. Compositio Math. **13** (1956), 39–46.

Définition des anneaux locaux d'une classe projective, et notion de classe localement normale. Pour toute classe C, la classe $i(C^h)$ est localement normale pour h assez grand. Caractérisations des classes localement normales. Une classe C est dite arithmétiquement normale si $C^t=i(C^t)$ pour tout $t\geqq 1$; elle est alors localement normale; pour toute classe C', la classe $i(C'^h)$ est arithmétiquement normale pour h assez grand. *P. Samuel.*

Gröbner, Wolfgang. Sopra lo scioglimento delle singolarità delle varietà algebriche. Boll. Un. Mat. Ital. (3) **11** (1956), 319–327.

Let $\mathfrak{d}_x=(p_1, p_2, \cdots, p_s)$, $p_j(x)\in K[x_0, \cdots, x_m]$, $j=1, \cdots, s$; $s\geqq 2$, be an homogeneous ideal, where the $x_0, \cdots, x_m$ are indeterminates over the field of complex numbers K. The author continues the study of the monoidal correspondence $(*)\ \rho y_{ij}=x_i p_j(x)$, $i=0, \cdots, m$; $j=1, \cdots, s$, that he had initiated in Monatsh. Math. **58** (1954), 266–286 [MR **16**, 741]. The author states: a) One can assume that all the forms $p_i(x)$, $i=1, \cdots, s$, are of the same degree. b) It is verified for the inverse correspondence of $(*)$ that: $(**)\ \rho x_i=y_{ij}$, $i=0, \cdots, m$; $j=1, \cdots, s$. c) The correspondence $(*)$ has as original variety the m-dimensional projective space S_m and $V(\mathfrak{v}_y)$ as image variety, where $\mathfrak{v}_y=K[y](y_{ij}y_{kl}-y_{il}y_{kj},\ y_{ij}p_\sigma(y_{kl})-y_{i\sigma}p_j(y_{kl})\cdots)$. d) The points of $V(\mathfrak{d}_x)$ have not any homologous point by $(*)$. e) To every point of $V(\mathfrak{v}_y)$ corresponds a well determined point of S_m. f) The correspondence $V(\mathfrak{v}_y)\to S_m$ is one to one on the whole $V(\mathfrak{v}_y)$, except for the points of the subvariety $V(\mathfrak{d}_y)$, where $\mathfrak{d}_y=(\mathfrak{v}_y,\ p_1(y_{i1}), \cdots, p_s(y_{is}))$. g) There is a one to one correspondence between the points of $V(\mathfrak{d}_y)$ and the tangential spaces to $V(\mathfrak{d}_x)$. h) If $V(\mathfrak{d}_x)$ has no singularities, it follows that $V(\mathfrak{v}_y)$ has no singularities. i) If $V(\mathfrak{d}_x)$ is the singular subvariety of an irreducible variety $V(\mathfrak{p}_x)$, the correspondence $((*), (**))$ transforms $V(\mathfrak{p}_x)$ into a variety $V(\mathfrak{p}_y)$ that generally, has no singularities.

{The reviewer must add to the observations made abovet he following ones: The correspondence $(*)$ is not an irreducible one, as the author implicitly assumes, since the bihomogeneous ideal $I=K[x;y](y_{ij}x_k p_l(x)-y_{kl}x_i p_j(x)$, $i, k=0, \cdots, m$; $j, l=1, \cdots, s$, corresponding to $(*)$, and obtained by eliminationof the parameter ρ among the $(*)$, is not a prime one. In order to obtain a non-trivial irreducible correspondence one must add to I certain polynomials and the following ideal follows

$$\mathfrak{B}=K[x;y](y_{ij}x_k p_l(x)-y_{kl}x_i p_j(x),$$
$$y_{ij}x_k-y_{kj}x_i,\ y_{ij}p_l-y_{il}p_j, \cdots)$$

In this ideal the polynomials $p_i(x)$ do not occur and hence the ideal $\mathfrak{d}_x$ is not transformed into an irrelevant ideal, in contradiction with d). The equations $(**)$ are true for the correspondence $\mathfrak{B}$, but not for $(*)$ which is in contradiction with b). If $\mathfrak{d}_x$ is a prime ideal, the proposition a) does not hold in every case, and if $\mathfrak{d}_x$ is not a prime ideal, some of the polynomials p_i can be reducible and this can effect essentially the basis of $\mathfrak{B}$ and the proof of g) by the author. The proposition e) is wrong, since there are fundamental subvarieties on both varieties S_m and $V(\mathfrak{v}_y)$. The author doesn't prove the proposition i) but his reasoning purporting to justify its validity uses the false proposition d).} *P. Abellanas* (Madrid).

Svoboda, Karel. L'hypersurface cubique à courbe double du quatrième degré dans un espace à quatre dimensions. Acta Acad. Sci. Čechoslovenicae Basis Brunensis **27** (1955), 393–404. (Czech. Russian and French summaries)

Dans ce mémoire on déduit quelques propriétés polaires de l'hypersurface cubique J, plongé dans un espace projectif à quatre dimensions, qui a une courbe rationelle normale C du quatrième degré pour courbe double et qui ce trouve engendrée par des bisecantes de cette courbe. L'hypersurface de Hesse, associée à l'hypersurface J, est formée de deux hypersurfaces, à savoir de J et de l'hypersurface quadratique I, qui fait la quadrique fondamentale de la correlation involutive determinée par la courbe C. L'hypersurface polaire d'une droite est un cône quadratique de la deuxième espèce qui est tangent en trois points à l'hypersurface J et en deux points à l'hypersurface I. L'hypersurface polaire d'un plan est une hypersurface cubique qui est tangente à l'hypersurface J le long d'une courbe du sixième degré et à l'hypersurface I le long d'une courbe du quatrième degré. L'hypersurface en question a une courbe double du quatrième degré qui touche l'hypersurface J en six points et l'hypersurface I en quatre points. L'hypersurface polaire d'un espace est une hypersurface biquadratique qui est tangente à l'hypersurface J le long d'une surface du sixième degré et à l'hypersurface I le long d'une surface du quatrième degré. L'hypersurface en question a une courbe double qui est formée d'une courbe du sixième degré, située sur l'hypersurface J, et d'une courbe du quatrième degré, située sur l'hypersurface I. (Résumé de l'auteur.) Ces recherches sont fondées sur les travaux de L. Seifert [Publ. Fac. Sci. Univ. Masaryk nos. **233**, **235** (1937)] et sur le livre de H. G. Telling, The rational quartic curve in space of three and four dimensions [Cambridge, 1936].

H. Schwerdtfeger (Victoria).

Barsotti, Iacopo. Gli endomorfismi delle varietà abeliane su corpi di caratteristica positiva. Ann. Scuola Norm. Sup. Pisa (3) **10** (1956), 1–24.

The author associates to an abelian variety over an algebraically closed field K of characteristic $p>0$ the formal (commutative) Lie group which is naturally obtained by considering the group law of the variety as operating in the local ring at the origin, and applies to these groups the general theory developed by the re-

viewer. In a previous paper [Rend. Circ. Palermo (2) **5** (1956), 145–169] he has proved in particular that if p^f is the order of the kernel of the endomorphism $p \cdot \delta_A$ of the abelian variety A of dimension n, then f is the dimension of the core of the corresponding formal Lie group. He now considers two abelian varieties A, B over K, and a homomorphism π of A; let p^h be the order of the kernel of $p \cdot \delta_B$; then to π is associated, in addition to the p-adic matrix $M_p(\pi)$ of type $h \times f$ defined in Weil's "Variétés abéliennes et courbes algébriques" [Hermann, Paris, 1948; MR **18**, 621], another p-adic matrix $M_{p'}(\pi)$ of the same type, which corresponds to the homomorphism of the core of A into the core of B, defined by π; in the special case $B=A$, $f=n$, the product of the characteristic polynomials of $M_p(\pi)$ and $M_{p'}(\pi)$ is the characteristic polynomial of π. Similarly, the author also defines, for every divisor X of A, an $f \times f$ matrix $E_p(X)$, which has properties similar to those of the matrices $E_l(X)$ defined in § XI of "Variétés abéliennes et courbes algébriques" for primes $l \neq p$. Finally, if A is not special (i.e. if there are no exact differentials of the first kind on A except 0), if $\mathfrak{A}_0(A)$ is the ring of its endomorphisms, and $\mathfrak{A}(A)$ the simple algebra over the rationals obtained from $\mathfrak{A}_0(A)$ by tensor product with the rationals, then for any maximal order $\mathfrak{o}$ of $\mathfrak{A}(A)$, there is an abelian variety A' isogenous to A such that $\mathfrak{A}_0(A')$ is transformed from the order $\mathfrak{o}$ by an inner automorphism of $\mathfrak{A}(A)$. However, there are in general abelian varieties isogenous to A and whose ring of endomorphisms is not a maximal order of $\mathfrak{A}(A)$. *J. Dieudonné* (Evanston, Ill.).

Rosenlicht, Maxwell. Some basic theorems on algebraic groups. Amer. J. Math. **78** (1956), 401–443.

The purpose of the paper is to give a systematic development of some basic results in the theory of algebraic groups. The main results proved are structure theorems dealing with the existence in an algebraic group G of a normal algebraic subgroup H such that H and G/H are of specified types. The most important of these results, announced by Chevalley in 1953 and first proved in print by Barsotti [Ann. Mat. Pura Appl. (4), **38** (1955), 77–119; Atti Accad. Naz. Lindei. Rend. Cl. Sci. Fis. Mat. Nat. **18** (1955), 43–50; MR **17**, 193, 897], asserts that if G is connected then there exists a linear H such that G/H is an abelian variety. The first two parts of the paper develop the necessary concept of factor group, in its natural setting of transformation spaces and orbit varieties. The general methods stem from Nakano [Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27** (1952), 55–66; MR **14**, 201] and Weil [Amer. J. Math. **77** (1955), 355–391, 493–512; MR **17**, 533] but here are generalized to permit reducible varieties. The third part deals with the homomorphism theorems. The natural group isomorphism $G/H \approx (G/N)/(H/N)$ is shown to be birational, but the natural isomorphism $H/(H \cap N) \approx HN/N$ is not always so; it is, however, rational and purely inseparable. Reformulating Weil's notion of "isogeny" of abelian varieties in a way appropriate to arbitrary algebraic groups, the author obtains a result somewhat more precise than the fact that $H/(H \cap N) \approx HN/N$ is a purely inseparable isogeny. This permits him similarly to recapture for algebraic groups the Jordan-Hölder-Schreier theorem via Zassenhaus' lemma. The fourth part treats connected algebraic groups which are solvable (in the strong sense that the factor groups of some normal chain are all birationally isomorphic to either the additive or multiplicative group of the universal field. The main result here is that if V is a transformation space for such a group G and if $\tau: V \to W$ denotes the canonical generically surjective rational mapping of V onto the orbit variety W, then there exists a "cross-section", i.e. a rational mapping $\sigma: W \to V$ such that $\tau\sigma = 1$. The final part of the paper develops the structure theorems as mentioned above.

E. R. Kolchin (New York, N.Y.).

Barsotti, Iacopo. Algebraic group-varieties. Bull. Amer. Math. Soc. **62** (1956), 519–530.

This paper is the substance of an expository address delivered in June 1955. After a brief account of the genesis of the theory of group varieties in classical analysis, and of the subsequent freeing of the theory from analytic considerations made possible by the emergence of abstract algebraic geometry, the author describes the main results of the modern (as of June 1955) theory, due to himself, Weil, Chevalley, Rosenlicht, and others, with emphasis on his own contributions. *E. R. Kolchin.*

See also: Yamazaki, p. 468; Seidenberg, p. 469; Marmion, p. 500; Norden, p. 502.

NUMERICAL ANALYSIS

Numerical Methods

Householder, Alston S. On the convergence of matrix iterations. J. Assoc. Comput. Mach. **3** (1956), 314–324.

The author considers iterative solutions for the system $Ax=h$, of the type $x_{\nu+1}=k+Cx_\nu$. A is a square irreducible matrix, and in the major part of the paper, the elements on the main diagonal are positive. Particular attention is paid to the relation between convergence conditions (necessary and sufficient) for total- and single-step iterations. It is shown that diverse known results can be elegantly obtained by the method of matric norms.

E. Reich (Minneapolis, Minn.).

Morton, K. W. On the treatment of Monte Carlo methods in text books. Math. Tables Aids Comput. **10** (1956), 223–224.

Beale, E. M. L. Cycling in the dual simplex algorithm. Naval Res. Logist. Quart. **2** (1955), 269–275 (1956).

In principle, if there exist feasible bases in the simplex method of solution of a linear programming problem for which two or more of the associated coefficients are zero, then the steps of the method may cycle. This has never occurred in practice and there exist simple devices (at least theoretically) for preventing this by slight modification of the simplex algorithm. Nevertheless, interest in the phenomenon of cycling persists; and since the "deep reason", if there is any, why it may happen is still unknown, the author's example will be welcome to those still curious about the phenomenon. The only other example in the "literature" (privately circulated in 1952) is due to the reviewer, and it is much easier to see the cycle happen (i.e. the geometric motivation) in the present example. In both examples, the number of zero coefficients associated with basis vectors is constantly two, and the author's method permits the deduction of two

theorems about this case: that the number of steps in the cycle is always even, and at least six. *A. J. Hoffman.*

Wagner, Harvey M. A two-phase method for the simplex tableau. Operations Res. 4 (1956), 443–447.

Most of this paper is devoted to an exposition of the simplex method and the use of artificial vectors for finding a feasible solution. For the case in which the feasible solution is found with artificial vectors still in the basis, but whose associated coefficients are zero, the author advocates consideration of the "true" objective function only, in subsequent iterations, and prescribes rules for doing this. *A. J. Hoffman* (New York, N.Y.).

Slater, L. J. The real zeros of the confluent hypergeometric function. Proc. Cambridge Philos. Soc. **52** (1956), 626–635.

The author gives a series expansion for zeros of ${}_1F_1(a, b; x)$ qua function of x, in terms of a first approximation and the correction by Newton's method to this first approximation. She gives corresponding expansions for the derivative of ${}_1F_1$ at a zero, for turning points (zeros of the derivative), and for the value of ${}_1F_1$ at a turning point. An alternative method for computing these quantities on an automatic digital computer, the computation of zeros of ${}_1F_1$ qua function of a and qua function of b are also described. Diagrams show the curves in the a, b plane along which ${}_1F_1(a, b; 1)$ and ${}_1F_1(a, b; 2)$ vanish, and a 7D table of the first positive zero of ${}_1F_1(a, b; x)$ qua function of x is given for

$$a=-4(.2)-2(.1)-.1 \text{ and } b=.1(.1)1.2(.2)2.4.$$

Some unpublished tables and computations of the author are also described.

A. Erdélyi (Jerusalem).

Fiedler, Miroslav. Über das Gräffesche Verfahren. Czechoslovak Math. J. **5(80)** (1955), 506–516. (Russian summary)

The author considers the Graeffe process as amplified by Brodetsky and Smeal [Proc. Cambridge Philos. Soc. **22** (1924), 83–87] and the reviewer [Math. Tables Aids Comput. **1** (1945), 377–383; MR **7**, 84]. The Graeffe product $F \circ G$ of two polynomials F and G is defined as the polynomial H such that

$$2H(x^2)=F(x)G(-x)+G(x)F(-x).$$

Given a polynomial f of degree n whose roots are required, two sequences f_k, g_k of polynomials are generated recursively by means of the relations

$$f_0=(-1)^n f,\ g_0=nxf-x^2f',$$

$$f_{k+1}=f_k \circ f_k,\ g_{k+1}=f_k \circ g_k.$$

It is in terms of the coefficients of f_k and g_k that precise information about the roots of f is obtained. In particular, f_k is the original kth iterate of Graeffe and has for its roots the 2^kth powers of the roots of f. Information from g_k enables one to determine the roots of f without ambiguity. Three theorems are proved in the difficult general case of any complex roots. The process is applied to the case in which f is the characteristic polynomial of an n by n matrix. Typographic complexities prevent an adequate presentation of the results in this review.

D. H. Lehmer (Berkeley, Calif.).

Krylov, V. I. Approximate evaluation of integrals in the case of rapidly oscillating factors in the integrand. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 1014–1017. (Russian)

A function $g(x)$ is called "rapidly oscillating about 0 on $[a, b]$" if (i) $\int_a^b g(x)dx=0$; (ii) $[a, b]$ is the union of many short intervals d_i for which $\int_{d_i} g(x)dx=0$; and (iii) $g(x)$ does not deviate far from 0 (in some norm). Given such a function $g(x)$ and a slowly changing $f(x)$ with several derivatives, the problem is to use numerical quadrature to evaluate $I=\int_a^b f(x)g(x)dx$ without demanding too fine a subdivision. An answer is found by repeated partial integration, developing I into an analog of the Euler-Maclaurin formula. A remainder term, involving the nth derivative of f and the nth integral of g, is bounded in terms of $\int_a^b (f^{(n)})^2 dx$ and the Fourier coefficients of g. [For related work see Erugin and Sobolev, Prikl. Mat. Meh. **14** (1950), 193–196; MR **11**, 717.]

G. E. Forsythe (Los Angeles, Calif.).

Plainevaux, J. E. Détermination graphique directe de la valeur moyenne d'une fonction. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 939–949.

L'auteur expose un procédé graphique qui permet de trouver:

$$\mu(x)=x^{-1}\int_0^x f(x)\,dx$$

sans passer par la détermination préalable de $\int_0^x f(x)dx$. Il est possible de déterminer également la tangente à la courbe $\mu(x)$ et son centre de courbure. Les résultats d'un essai pratique sont donnés. *J. Kuntzmann* (Grenoble).

Longman, I. M. Note on a method for computing infinite integrals of oscillatory functions. Proc. Cambridge Philos. Soc. **52** (1956), 764–768.

This note concerns the numerical evaluation of integrals with an oscillating integrand and with an infinite upper limit. The author evaluates the integral over the intervals between consecutive zeros of the integrand and obtains in this way an alternating series which is summed by aid of Euler's transformation. In a modification of this method points near to the turning points of the integrand are used instead of the zeros. Two numerical examples show the feasibility of the method. *D. J. Hofsommer.*

Aoyama, Hirojiro. On the evaluation of the sampling error of a certain determinant. Ann. Inst. Statist. Math., Tokyo **8** (1956), 27–33.

The author estimates the error in the value of a determinant which is produced by errors in measuring the elements. He assumes that the elements are real non-negative independent random variables with equal variances. He obtains an upper bound for the variance of the determinant. His results compare favorably with a similar estimate made by Hadamard. *A. H. Copeland.*

Ergin, E. I. Transient response of a nonlinear system by a bilinear approximation method. J. Appl. Mech. **23** (1956), 635–641.

The author considers the differential equation $x''+g(x)=F(t)$. The nonlinear term $g(x)$ is approximated by a continuous, piecewise linear function $g_1(x)$, given by k_1x if $x \leqq x_t$, and by $k_2x+(k_1-k_2)x_t$ if $x \geqq x_t$. The slope k_1 is taken as $g'(0)$. The maximum amplitude, x_m, of the solution is estimated. The transition amplitude, x_t, and the slope k_2 are then chosen to minimize the mean square

difference between $g(x)$ and $g_1(x)$ on the range $0=x=x_m$. It is shown that if $g(x)=k_1x+\varepsilon x^n$ on this range, then x_t is approximately $x_m(n-1)/(n+2)$.

The method is studied for $g(x)=k_1x+\varepsilon x^3$ with various forcing terms, $F(t)$. The results are compared with solutions found by numerical integration, and also with results obtained by the method of Kryloff and Bogoliuboff [Introduction to non-linear mechanics, Princeton, 1943; MR 4, 142] and the method of equivalent linearization [Minorsky, Introduction to non-linear mechanics, Edwards, Ann Arbor, Mich., 1947; MR 8, 583]. The present method is shown to give closer approximations in the cases studied. It is shown that the accuracy of solutions is not very sensitive to errors in estimating the amplitude x_m. *W. S. Loud* (Minneapolis, Minn.).

Clenshaw, C. W. The numerical solution of linear differential equations in Chebyshev series. Proc. Cambridge Philos. Soc. **53** (1957), 134–149.

This interesting and useful paper describes an extension and improvement of the methods developed by C. Lanczos for solving differential equations.

Linear differential equations, preferably with polynomial coefficients but with any type and position of associated linear boundary conditions, are solved by considering Chebyshev expansions not only for the wanted function but also for those of its derivatives which occur in the equation. Simple integral relations between the coefficients in the various expansions, and extra relations derived from the differential equation and boundary conditions, form a simple system of linear simultaneous algebraic equations, effectively infinite in number, but made finite by suitable truncation of the Chebyshev series. The paper describes methods for solving the algebraic equations, and illustrates techniques with several worked examples, including a second-order equation of eigenvalue type, and a case in which Taylor's series is divergent. *L. Fox* (Berkeley, Calif.).

Hersch, Joseph; Pfluger, Albert; und Schopf, Andreas. Über ein simultanes Differenzenverfahren zur Abschätzung der Torsionssteifigkeit und der Kapazität nach beiden Seiten. Z. Angew. Math. Phys. **7** (1956), 89–113.

Pólya [e.g., pp. 200–207 of von Mises Anniversary Volume, Academic Press, New York, 1954; MR **16**, 482] showed how to use finite difference equations together with a minimum (or maximum) principle to obtain upper (or lower) bounds for certain domain functionals. The present paper is the first major exposition of some of the authors' announced ideas for extending the use of finite difference equations to yield two-sided bounds for these functionals. [See, e.g., Hersch, C. R. Acad. Sci. Paris **240** (1955), 1602–1604; MR **16**, 929.]

Let $u_{xx}+u_{yy}=-2$ in a plane region Q assumed to be the union of squares of a square net, with $u=0$ on the boundary Γ of Q. For any scalar $f(x, y)$, define

$$\bar{f}=\iint_Q f\,dx\,dy \text{ and } D(f)=\iint_Q (f_x^2+f_y^2)\,dx\,dy.$$

The authors seek bounds for $P=2\bar{u}=D(u)$, the torsional rigidity of a prism with cross-section Q. Let ϕ be any piecewise smooth function vanishing on Γ. Extending Dirichlet's principle, Pólya and Szegö [Isoperimetric inequalities in mathematical physics, Princeton, 1951; MR **13**, 270] proved $\max_\phi P^-(\phi)=P$, where $P^-(\phi)=4\bar{\phi}^2/D(\phi)$. It was Pólya's idea [loc. cit.] to define a test function ϕ by its values ϕ_t at the nodes of the net, with bilinear interpolation in each square of the net. Maximizing $P^-(\phi)$ over all choices of the ϕ_t requires solving a certain set $\mathfrak{S}^-$ of linear difference equations and yields a lower bound for P.

Upper bounds are obtained from Thomsen's principle of minimal potential energy. Related ideas have been exploited by Synge [pp. 141–165 of Proc. Symposia Appl. Math., v. IV, McGraw-Hill, New York, 1954; MR **15**, 257]. Let Q be divided by smooth curves Γ_t into subregions. Define a piecewise smooth vector field $\mathbf{v}=(v_x, v_y)$ in Q so that $\operatorname{div}\mathbf{v}=-2$, and so that the normal component of $\mathbf{v}$ is continuous at each Γ_t. Then

$$\min_{\mathbf{v}} P^+(\mathbf{v})=P,$$

where $P^+(\mathbf{v})=\iint_Q (v_x^2+v_y^2)\,dx\,dy$. Now define a trial field $\mathbf{v}$ by its (constant) normal components S_t at the sides of the square net, with linear interpolation of both v_x and v_y within each square. Minimizing $P^+(\mathbf{v})$ over all choices of the S_t requires solving a set $\mathfrak{S}^+$ of linear difference equations and yields an upper bound for P. Now $\mathfrak{S}^+\neq\mathfrak{S}^-$.

The author's most interesting contribution here is to show how to obtain a lower bound $P_2^-<P^-$ out of the solution of $\mathfrak{S}^+$. Details are too complicated to give here. Thus solving one linear system $\mathfrak{S}^+$ can yield the bounds $P_2^-<P<P^+$. The various bounds are computed numerically for $h=1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ for a symmetric cross Q which is the union of five unit squares.

Other bounds are also derived. The ideas are also applied to bound the electrostatic capacity of a condensor in 2 and 3 dimensions. *G. E. Forsythe.*

Legendre, Robert. Calcul approximatif d'une grille d'aubes transsonique. Rech. Aéro. no. **53** (1956), 3–12.

This paper presents a method for construction of approximate solutions for a lattice in the hodograph plane. In certain particular case, it leads to exact solutions in series form whose convergence is established only in the low speed range. *Y. H. Kuo* (Peking).

See also: Betts, Bhatia and Wyman, p. 475; Langenbah, p. 475; Mitrinović, p. 477; Wynn, p. 478; Mikeladze, p. 479; Seeger, p. 489; Prinz, p. 517; Lah, p. 520; Kopzon, p. 526; Gold, p. 526; Spence, p. 530; Uchida, p. 530; Baron, p. 531; Krzywoblocki, p. 532; Fowell, p. 532; Wu, p. 536; Beale, p. 514.

Graphical Methods, Nomography

Deuel, P. D. A nomogram for factor analysts. Psychometrika **21** (1956), 291–294.

When a new reference vector is chosen graphically from the plane of two old ones, its direction cosines as well as the projections of the tests on it are most easily computed by applying certain multipliers d and Sd to quantities which are already known. The nomogram quickly supplies d, after S has been read from the graph. (Author's summary.) *R. L. Anderson* (Raleigh, N.C.).

See also: Kegel, p. 460; Atkinson, p. 484; Štepa, p. 535; Millsaps and Pohlhausen, p. 538.

Tables

★ **Triebel, Franz. Rechen-Resultate. Achte Auflage.** Technischer Verlag Herbert Cram, Berlin, 1956. ii+285 pp. DM 26.00.

This gives, on page $n+2$ for $n=1(1)100$, the product

mn to two decimal places, where $m=1(1/4)100$. There follows the product mn for $m=101(1)1000$, $n=1(1)100$, each page covering five values of m; there is also a table of mn, $m=1(1)300$, $n=1/6$, $1/3$, $2/3$.

There is a brief introduction showing the use of the tables. There are elaborate thumb indices. Printing is satisfactory; the accuracy has not been checked.

J. Todd (Washington, D.C.).

★**Schmidt, Walter. Der Rechner.** Technischer Verlag Herbert Cram, Berlin, 1955. xix+200 pp. DM 18.00.

This gives on page n, $n=1(1)200$, the product mn, to one place of decimals, where $m=p+\theta$, $p=0(1)100$, $\theta=0.1(.1).9$; 1/6, 1/3, 2/3, 5/6; 1/4, 3/4. The format is satisfactory and the printing tolerable. The accuracy has not been checked.

There is a rather light-hearted introduction, which gives various examples of the use of the table. *J. Todd.*

García, Juan. New tables of logarithms. Las Ciencias **19** (1954), 567–592. (Spanish)

Some new "triple-entry" arrangements of tables of logarithms of numbers and trigonometrical functions are described by means of specimen pages and examples. For instance, the tables of logarithms gives logarithms of numbers increasing by 2 from 100 000. Values of 4 figure argument such as 1234 can be read off directly while that of an argument such as 123456 is obtained by adding the contribution corresponding to the increment 56, which is read off from a table superimposed on the main table. {Suppression of leading figures contributes to compactness and confusion.} *J. Todd* (Washington, D.C.).

★**Tables of the cumulative binomial probability distribution.** By the Staff of the Computation Laboratory. The Annals of the Computation Laboratory of Harvard University, vol. **35.** Harvard University Press, Cambridge, Mass., 1955. lxi + 503 pp. $8.00.

The function described in the title is defined more succinctly by the notation

$$E(n, r, p)=\sum_{x=r}^{n} p^x \binom{n}{x}(1-p)^{n-x}.$$

The function is tabulated to 5 decimals for

$$n=1(1)50\ (2)100\ (10)200\ (20)500\ (50)1000,$$

for $r=0(1)n$ and for $p=.01(.01).5$ as well as for ten additional rational values of the form $k/16$ and $k/12$ and less than $1/2$. The table is left blank when $E<5.10^{-6}$, or when E is correspondingly close to unity. In spite of this the table is quite extensive. It is arranged in six parts each devoted to values of p in an interval of length $1/12$. An appendix gives the common logarithms of $n!$ for $n=0(1)1199$ to 10 decimals.

There is an extensive introduction covering the troublesome problem of interpolation in this 3-way table and a fine 28 page account of applications by F. Mosteller.

This very useful table surpasses all previous tables of this function in its range of n and, though not as accurate as earlier ones, is sufficient for most applications in statistics.

This table was the first produced by Harvard's Mark IV by what must have been an elaborate set of programs. The format and printing are superlative.

D. H. Lehmer (Berkeley, Calif.).

Chaudhuri, S. B. Statistical tables and certain recurrence relations connected with p-statistics. Calcutta Statist. Assoc. Bull. **6** (1956), 181–188.

Percentage points of the distribution of a certain function of the p-statistics (the p roots of the determinantal equation $|A-\lambda B|=0$, where A, B are sample covariance matrices from two p-variate normal popolations) are calculated for $p=2$, 3 and certain sample sizes.

H. Teicher (Lafayette, Ind.).

See also: Oberhettinger, p. 481; Slater, p. 515; David, p. 521; Fisher and Healy, p. 521; Hitchcock, p. 524; Millsaps and Pohlhausen, p. 538.

Machines and Modelling

★**Prinz, D. G. Some experiences on the Manchester computer with the Simplex method.** Conference on linear programming, May, 1954, pp. 73–89; discussion, 90–91. Ferranti Ltd., London.

This paper presents an account of the coding of the simplex method on the Manchester computer by the author. It is free from fussy details pertinent to a particular machine, but it does not shirk explanation of the numerical difficulties encountered by round of errors and how they were overcome. Mathematically, the most interesting point is the attempt to show some justification for the fact that the simplex method seems to take a number of iterations much lower than the conceivable (though never properly estimated) upper bound. The author arrives at the average result $\log_n C_m$ where m is the number of rows, n the number of columns of the tableau. The reasoning is heuristic and not particularly convincing, but it is, so far as the reviewer knows, the first brave effort to estimate the number of iterations.

A. J. Hoffman (New York, N.Y.).

Jeffrey, Richard C. Arithmetical analysis of digital computing nets. J. Assoc. Comput. Mach. **3** (1956), 360–375.

The author indicates several possible numerical interpretations of a sequence of binary digits, where the sequence may be temporal or spatial. They differ according to the assumed location of the radix point and the representation of negative numbers. He then shows how the Boolean algebraic description of a computing net can be translated into an arithmetic description of its operation in terms of whichever interpretation is of interest.

A. S. Householder (Oak Ridge, Tenn.).

Mason, Robert M. The digital approximation of contours. J. Assoc. Comput. Mach. **3** (1956), 355–359.

The problem is to program a computer to trace a contour $f(x, y)=k$ on a grid. Suppose, for definiteness, $f<k$ inside the contour line. The sign of $f-k$ is examined at a sequence of points differing by Δx or by Δy. The sequence is rectilinear until $f-k$ changes sign, after which the direction rotates by $\pi/2$ if $f-k$ has become positive, $-\pi/2$ if $f-k$ has become negative. For getting onto the contour initially the machine seeks out a path of steepest ascent or descent. *A. S. Householder.*

Kessler, K. G.; Prusch, S. B.; and Stegun, I. A. Analysis of atomic spectra with electronic digital computers. J. Opt. Soc. Amer. **46** (1956), 1043–1045.

A technique for utilizing electronic digital computers in

the analysis of atomic spectra has been developed. Observed wave length are first converted to vacuum wave numbers using Edlén's formula. A search and sort is then performed on the wave numbers to obtain all pairs of lines with allowable energy differences. The energy levels predicted by these lines are computed and form the card output of the program. Only energy levels predicted more than once are assumed to be real levels. An analysis of the arc spectrum of ruthenium was performed using the above technique on the SEAC at the National Bureau of Standards. *R. M. Davis* (College Park, Md.).

Morton, J. P. A programme for low aspect ratio wing analysis. A digital computer programme to analyse the structure of a low aspect ratio wing. Aircraft Engrg. **28** (1956), 415–418.

Akušskiĭ, I. Ya. Operational matrices of the difference scheme. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1956**, no. 4(8), 101–111. (Russian)

The programming of the computation of a difference table by a machine with a number of counters and an unspecified repertory of commands is discussed by means of an elaborate matrix representation. *D. H. Lehmer.*

Barnes, George. Hatchet or hacksaw blade planimeter. Amer. J. Phys. **25** (1957), 25–29.

Masuyama, Motosaburo. An elementary method of construction of punched cards for p^n- and other designs. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4** (1956), 78–84.

Avsi, Serge. "Le mélange brownien" en tant qu'origine d'une loi physique des fluctuations. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 861–872.

Examples of spatial counts in varied physical contexts with empirical χ^2/d.f. (computed under the assumption of uniform multi-Bernouilli trials) near .92. *J. Hannan.*

See also: Rohleder, p. 457; Atkinson, p. 484; Slater, p. 515; Ergin, p. 515; Suzuki, p. 522; Pearcey and Hill, p. 524.

PROBABILITY

Davidson, Donald; and Suppes, Patrick. A finitistic axiomatization of subjective probability and utility. Econometrica **24** (1956), 264–275.

The authors propose an axiom system for a finitistic rational choice structure, capable of an interpretation yielding a numerical utility function (unique up to a linear transformation) and an (absolutely unique) numerical subjective probability function. Difficulties of various sorts are pointed out in connection with previous proposals along these lines, particularly in case of measure derivations technically requiring for verification an examination of infinitely many alternatives. Any theory without behavioristic predictive force is inadequate for other than normative purposes. Identification of subjective and objective probability is here rejected, as without empirical justification. Basic terms are interpreted as a finite set of alternative choices in each act of decision, a finite set of distinct events, an ordered scale of preferences, a formalized indifference state, and an event with subjective probability one-half. From twelve postulates twelve theorems are deduced. *A. A. Bennett.*

Saxer, W. Beziehungen zwischen Statistik und Wahrscheinlichkeitsrechnung. Math.-Phys. Semesterber. **5** (1956), 49–62.

An expository account of the historical development and major concepts of mathematical statistics as an outgrowth of Laplace's theory of probability.

A. A. Bennett (Providence, R.I.).

Pfanzagl, Johann. Das Gesetz der grossen Zahl. Statist. Vierteljschr. **9** (1956), 16–21.

The law of large numbers is used in this paper as a designation for the statement that the success ratio of an event converges to the probability as the number of trials is indefinitely increased. The author remarks that there is still some tendency to regard this law as a consequence of the Bernouilli theorem. He points out some of the logical difficulties in so regarding it. He also presents a brief critique of various interpretations of the word "probability". *A. H. Copeland* (Ann Arbor, Mich.).

Wintner, Aurel. Cauchy's stable distributions and an "explicit formula" of Mellin. Amer. J. Math. **78** (1956), 819–861.

The paper is divided into five parts: a first section followed by four appendices. The first part deals with Cauchy's symmetric stable distributions $F_\beta(t)$ of index β, where $F_\beta(t)$ is defined by the property that its Fourier transform is even and equal to $\exp(-t^\beta)$ for $0 \leq t < \infty$, i.e. $F_\beta(t) = \pi^{-1} \int_0^\infty \exp(-s^\beta) \cos ts \, ds$. Three issues are considered, namely: "(i) the explicit form of the Laplace transform of the distribution, (ii) the connection of the Laplace transform with the Stieltjes transform, finally (iii) the application of the resulting explicit formulae to a determination of the weight function which belongs to the 'stratification,' in terms of the symmetric Gaussian distribution ($\beta=2$), of the symmetric stable distribution of any index $\beta<2$." Also in this first section there are considered certain asymptotic expansions in connection with the function $\phi(x) = \exp(-tx^\alpha)$ for arbitrary $\alpha>0$, and which correspond to an explicit evaluation (Mellin) of the Euler-MacLaurin difference $\sum_n^\infty{}_1 \phi(n) - \int_0^\infty \phi(x)\,dx$ in the case $0<\alpha<1$. An asymptotic formula valid for even α ($\alpha>2$, the case $\alpha=2$ being trivial) due to Pólya is here published for the first time.

Appendix I deals with the function

$$\theta_\alpha(x; q) = 1 + 2 \sum_{m=1}^{\infty} q^{m^\alpha} \cos 2\pi m x$$

($0<\alpha<\infty$, $0<q<1$, $-\infty<x<\infty$), which becomes Poisson's kernel if $\alpha=1$ and an elliptic theta function if $\alpha=2$. The singularity of this function at $x=0$ (when $0<\alpha<1$, $0<q<1$) is discussed. Also there is obtained a formula whereby $\theta_\alpha(x; q)$, being interpreted as a density of an angular distribution, is represented as a "Gaussian stratification."

Among other things, it is shown in Appendix II that every symmetric L-distribution (Khintchine-Lévy) is convex. (A convex distribution is defined as a distribution which for $|x|>0$ has a density, not necessarily continuous, which is a monotone function of $|x|$.) Appendix III deals with symmetric L-distributions. Appendix IV concerns angular distribution densities and in particular the func-

tion $\theta(q;\phi)=1+2\sum_{m=1}^{\infty} q^{m^2}\cos m\phi$ (the case $\alpha=2$ of the function θ_α considered before). Another proof is given here of an earlier result of the author, namely that the function $f(\phi)=\theta(q;\phi)$ is bell-shaped (mod 2π) for every fixed q satisfying $0<q<1$. The proof depends upon the interesting general lemma: If $a_1, a_2, \cdots$ is a sequence of numbers satisfying $0\leq a_n\leq 1$ and $\sum a_n<\infty$ then the function $\prod(1+a_n\cos\phi)$ is bell-shaped (mod 2π).

C. R. Putnam (Lafayatte, Ind.).

★**Fortet, Robert. Normalverteilte Zufallselemente in Banachschen Räumen. Anwendungen auf zufällige Funktionen.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 29–35. Deutscher Verlag der Wissenschaften, Berlin, 1956.

Expository lecture. *J. L. Doob* (Geneva).

Lévy, Paul. Le dernier manuscrit inédit de W. Doeblin. Bull. Sci. Math. (2) **80** (1956), 61–64.

The author reports on a manuscript left in an incomplete form by Doeblin, in which conditions on an aggregate of distribution classes are indicated sufficient that the aggregate be the set of all possible limit classes of the distribution classes of sums of independent random variables with a common distribution function. The manuscript has never been completely clarified.

J. L. Doob (Geneva).

Kawata, Tatsuo. Typical functions of sums of non-negative independent random variables. Kōdai Math. Sem. Rep. **8** (1956), 13–22.

Let $X_1, X_2, \cdots$ be mutually independent non-negative uniformly integrable random variables, with respective distribution functions $F_1, F_2, \cdots$, let $S_n=\sum_1^n X_k$, and let $\Phi\sigma_n$ be the expected value of $h^2/(S_n^2+h^2)$. It is supposed that $0<E\{X_i\}$, that $E\{X_n\}\to m$ in the sense of Cesaro, and that $F_i(x)-F_i(0)\leq Ax^p$, for $0<x<\delta$, where δ, A, p do not depend on i and $p>1$. It is proved that then $\lim_{h\to\infty}\sum_1^\infty\Phi_{\sigma_n}(h)/h=\pi/(2m)$. Furthermore, if $N(h)$ is integral-valued, and if $N(h)/h\to\infty$, this conclusion remains true if the upper summation limit is replaced by $N(h)$. Finally, it is shown that, for any positive h,

$$\lim_{n\to\infty}\Phi_{\sigma_n}(nh)=h^2/(m^2+h^2).$$

J. L. Doob (Geneva).

★**Kolmogoroff, Andrej; und Prochorow, Juri. Zufällige Funktionen und Grenzverteilungssätze.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 113–126. Deutscher Verlag der Wissenschaften, Berlin, 1956.

An amalgam of two expository lectures. Of particular interest is the treatment of convergence of distributions, which goes from the most abstract definition to applications to specific probability limit theorems.

J. L. Doob (Geneva).

van Elteren, Ph. The asymptotic distribution for large m of Terpstra's statistic for the problem of m rankings. Math. Centrum Amsterdam. Statist. Afdeling Rep. S 212 (VP 12) (1956), 14 pp.

Let $x^{(\alpha)}=(x_1^{(\alpha)}, \cdots, x_n^{(\alpha)})$ $(\alpha=1, \cdots, m)$ be m random vectors with n components, let

$$T_{\alpha,\beta}=\sum_{i<j}\operatorname{sgn}(x_i^{(\alpha)}-x_j^{(\alpha)})\operatorname{sgn}(x_i^{(\beta)}-x_j^{(\beta)}),$$

and $T=\sum_{\alpha<\beta}T_{\alpha,\beta}$. The author considers the distribution of T when the m vectors are independent and identically distributed, and the numbers of "tied" components in each vector are held fixed. For n fixed and $m\to\infty$, T is found to be asymptotically distributed as a linear combination of two independent chi squares. A comparison with the exact distribution for certain m and n is made. The case of missing observations is also considered. The asymptotic distribution of T for m fixed and $n\to\infty$ was derived by Terpstra [Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955), 690–696; **59** (1956), 59–66; MR **17**, 983].

W. Hoeffding (Chapel Hill, N.C.).

Derman, Cyrus. Some asymptotic distribution theory for Markov chains with a denumerable number of states. Biometrika **43** (1956), 285–294.

Let C be a Markov chain with non-negative integer times, stationary transition probabilities, a countable number of states, all belonging to the same class and all ergodic, and let the second moments of the recurrence times be finite. Using the central limit theorem of Doeblin [Bull. Soc. Math. France **66** (1938), 210–220], the author shows that, with a suitable scaling, the joint distribution of the empirical frequencies of runs of several different sorts tends, as the number of trials increases, to a Gaussian distribution which he computes and uses to construct a goodness of fit test for the transition probabilities [see, also, M. S. Bartlett, Proc. Cambridge Philos. Soc. **47** (1951), 86–95; Biometrika **39** (1952), 118–121; MR **12**, 512; **13**, 962].

H. P. McKean, Jr.

Mabboux-Tariel, Genevieve; et Mabboux, Claude. Spectre énérgetique de certaines fonctions aléatoires de la forme ± 1. C. R. Acad. Sci. Paris **243** (1956), 1509–1511.

The spectral density of a random-function which is alternately $+1$ and -1 (telegraph signal) with the intervals between changes independent random variables with a common distribution is found in the form

$$A_v=\frac{n}{2\pi^2v^2}\left[\frac{1-\varphi(2\pi v)}{1+\varphi(2\pi v)}+\frac{1-\varphi^*(2\pi v)}{1+\varphi^*(2\pi v)}\right]$$

with n the mean number of changes in unit time, $\varphi(u)$ the characteristic function of the interval variable, and $\varphi^*(u)$ its complex conjugate. In the simplest case of exponential distribution of the intervals this reduces to the known result $n(n^2+\pi^2v^2)^{-1}$ as given by G. W. Kenrick [Phil. Mag. (7) **7** (1929), 176–196]. A more complicated case in which the exponential distribution is shifted to the right by τ_0 (the interval must be at least τ_0) is also given, for its interest in counter problems. Three curves show the variation of the spectral density with changes in τ_0, including the limiting singularity when $n\tau_0$ approaches unity.

J. Riordan (New York, N.Y.).

Fisz, M. Realizations of some stochastic processes. Studia Math. **15** (1956), 359–364.

Let $\{x_t, t\in I_0\}$ be a stochastic process on the compact parameter interval I_0. It is supposed that

$$P\{\lim_{s\to t}x_s=x_t\}=1$$

on the parameter interval, and that, if $I=[a,b]$ and if $a(I)=P\{x_a\neq x_b\}$, then the upper Burkill integral of $a(\cdot)$ is finite over compact subintervals in the interior of I_0. It is proved that almost every sample function of the process is a step function, and that the expected value of the number of jumps in any such subinterval is finite.

J. L. Doob (Geneva).

Gani, J. The condition of regularity in simple Markov chains. Austral. J. Phys. **9** (1956), 387–393.

The author presents proofs of the known theorem that the sequence of iterates of a finite-dimensional stochastic matrix converges if and only if 1 is a simple latent root of the matrix and all other roots have modulus less than 1.

J. L. Doob (Geneva).

Cartwright, D. E.; and Longuet-Higgins, M. S. The statistical distribution of the maxima of a random function. Proc. Roy. Soc. London. Ser. A. **237** (1956), 212–232.

This paper studies the statistical distribution of the maximum values of a random function which is the sum of an infinite number of sine waves in random phase. The results are applied to sea waves and to the pitching and rolling motion of a ship. *Author's summary.*

Mycielski, Jerzy. On the distances between signals in the non-homogeneous Poisson stochastic process. Studia Math. **15** (1956), 300–313.

It is supposed that events are occurring in accordance with a nonhomogeneous Poisson process, with probability $a(t)h+o(h)$ that an event will occur in the interval $[t, t+h]$. Here $a(\cdot)$ is supposed non-negative and continuous, with $\int_0^\infty a(t)<\infty$. Let $n\theta_n(y)$ be the number of the first n time intervals between events which are of length $\leq y$, and define

$$S_n(y)=1-\frac{1}{n}\int_0^{\tau_n} a(x)\exp\Big(-\int_x^{x+y} a(t)dt\Big)dx,$$

where $\int_0^{\tau_n} a(t)dt=n$. It is proved that

$$|E\{\theta_n(y)\}-S_n(y)|<(2\pi n)^{-1/2}$$

and various conditions are found under which

$$\lim_{n\to\infty} S_n(y)$$

exists. For example, it suffices that $\lim_{x\to\infty}\int_x^{x+y} a(t)dt$ exist ($\leq\infty$), for some $y>0$. *J. L. Doob* (Geneva).

Lah, Ivo. Das Restglied der Taylorschen Reihe des Rentenbarwertes und einige Formeln des Zinsfussproblems für grosse Zinsspannungen. Skand. Aktuarietidskr. **38** (1955), 165–179.

An expression is obtained for the remainder term in a Taylor's series for approximating the present value of a life annuity from tabulated actuarial functions at a different interest rate. This is used to appraise the accuracy of various approximate formulas.

T. N. E. Greville (Washington, D.C.).

Eberl, W. Zur wahrscheinlichkeitstheoretischen Deutung gewisser Mannschaftswettkämpfe. Österreich. Ing.-Arch. **10** (1956), 148–154.

This paper contains a mathematical model of an athletic contest such as football. The model is a type of one-dimensional brownian motion, that is, motion toward and away from a given goal line. The author gives a definition of relative strengths of two opposing teams and method (using maximum likelihood) of estimating this ratio from actual play. He also gives a method of rank ordering a set of teams which have played against one another.

A. H. Copeland (Ann Arbor, Mich.).

Kazakov, I. E. Approximate probability analysis of the exactness of performance of essentially non-linear automatic systems. Avtomat. i Telemeh. **17** (1956), 385–409 (1 plate). (Russian)

A thorough exposition of a method developed independently and at about the same time by the author and R. C. Boonton [Proc. Symposium on Nonlinear Circuit Analysis, Polytech. Inst. Brooklyn, 1953, pp. 369–391; MR **16**, 1036]. Consider a nonlinear element f which is subjected to a random input $X(t)$ and let

$$X(t)=m_x(t)+X^0(t),$$

where $m_x(t)$ is the expectation of $X(t)$. The element f is "linearized" by replacing it with a linear element characterized by two coefficients k_0 and k_1 such that the output of f is approximately given by $Y(t)=k_0m_x(t)+k_1X^0(t)$; k_0 and k_1 are determined by the relations $k_0=m_y/m_x$, where m_y is the expectation of $Y(t)$, and $k_1=\sigma_y/\sigma_x$, where σ_y and σ_x are the variances of Y and X, respectively. Alternatively, k_0 and k_1 may be determined from the least squares condition $E(Y-k_0m_x-k_1X^0)^2=\min$. The author considers various types of nonlinear elements and tabulates k_0 and k_1 as functions of m_x and σ_x. Taking a very general type of nonlinear feedback control system, he shows how the variance of the output can be calculated and analyzes in detail two cases of practical interest.

L. A. Zadeh (New York, N.Y.).

See also: Postnikov, p. 468; Leader, p. 470; Bochner, p. 484; Hannan, p. 520; Karle and Hauptman, p. 524; Lifšic, p. 538; Leruste, p. 543; Downton, p. 547; Hetz, p. 548; Kuhtenko, p. 549.

STATISTICS

David, F. N.; and Johnson, N. L. Reciprocal Bernoulli and Poisson variables. Metron **18** (1956), no. 1–2, 77–81.

Castellano, Vittorio. Sulle variabili divise in intervalli e le correzioni della media aritmetica e della varianza. Statistica, Bologna **16** (1956), 151–186.

Hannan, E. J. Exact tests for serial correlation in vector processes. Proc. Cambridge Philos. Soc. **52** (1956), 482–487.

A generalisation of the methods of an earlier paper [Biometrika **42** (1955), 133–142; MR **16**, 1040] to vector Markov processes. The independence of successive vector variates $\mathbf{u}_t$ is tested by examining the canonical correlations between $\mathbf{u}_{2t}$ and $(\mathbf{u}_{2t-1}, \mathbf{u}_{2t+1})$. The method is also extended to provide an exact test for the serial correlation of a vector or residuals from regression. Bounds are obtained for the asymptotic efficiencies of the tests.

P. Whittle (Wellington).

Kudô, Akio. On the invariant multiple decision procedures. Bull. Math. Statist. **6** (1956), 57–68.

The author presents a unified treatment of the slippage tests developed by E. Paulson [Ann. Math. Statist. **23** (1952), 610–616; MR **14**, 569] and D. R. Truax [ibid. **24** (1953), 669–674; MR **15**, 727]. The proofs are based on a theorem giving a necessary condition for a multiple decision procedure to be optimal among certain classes of invariant decision functions. *J. L. Snell.*

Des Raj. Some estimators in sampling with varying probabilities without replacement. J. Amer. Statist. Assoc. **51** (1956), 269–284.

The problem considered is the estimation of the total value of a character for a finite population from a sample, when the units are selected with varying probabilities without replacement. Several unbiased estimates are proposed, and their variances and unbiased estimators of these variances are obtained. Some of the variance estimators are always positive, a property not possessed by some of their earlier competitors. The results for one stage sampling are extended to multistage designs.

S. W. Nash (Vancouver, B.C.).

Lawley, D. N. A general method for approximating to the distribution of likelihood ratio criteria. Biometrika **43** (1956), 295–303.

Suppose a sample of size n is from a population whose distribution depends on $p+q$ parameters $\theta_1, \theta_2, \cdots, \theta_p, \theta_{p+1}, \cdots, \theta_{p+q}$. Let H_0 be the statistical hypothesis that $\theta_{p+1}, \cdots, \theta_{p+q}$ have specified values $\theta_{p+1}^{(0)}, \cdots, \theta_{p+q}^{(0)}$, respectively. Let $L^{(r)}$ denote the result of maximizing the logarithm of the likelihood of the sample with respect to $\theta_1, \cdots, \theta_r$ with "true" values substituted for the remaining θ's. Then under certain regularity conditions on the population distribution (being essentially those under which maximum likelihood estimators have a limiting normal distribution as $n\to\infty$), the author shows that except for terms of order $1/n^2$,

$$2q(L^{(p+q)}-L^{(p)})/(q+\varepsilon_{p+q}-\varepsilon_q)$$

has the same moments (under H_0) as χ^2 with q degrees of freedom, where ε_{p+q} and ε_q are certain rational functions of the mean values (under H_0) of the first three derivatives of L with respect to the parameters $\theta_1, \cdots, \theta_{p+q}$. This result is applied to three examples. The first is testing the hypothesis that the mean of a normal population has a specified value while the variance is unknown (i.e. a nuisance parameter). The second is testing the hypothesis that k normal populations have equal (but unspecified) variances. The third is testing the hypothesis that the correlation coefficient of a normal bivariate distribution has a specified value, while the two variances are nuisance parameters.

S. S. Wilks (Princeton, N.J.).

David, H. A. The ranking of variances in normal populations. J. Amer. Statist. Assoc. **51** (1956), 621–626.

Analogues of the "gap-test" of Tukey [Biometrics **5** (1949), 99–114; MR **11**, 43] and a multiple decision procedure of Duncan [ibid. **11** (1955), 1–42; MR **16**, 842] for variances instead of means are described. Tables are given to facilitate the use of these procedures.

I. R. Savage (Stanford, Calif.).

★ **Weber, Erna. Das Rückschlussproblem in der biologischen Statistik.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 81–87. Deutscher Verlag der Wissenschaften, Berlin, 1956.

Wegner, L. H. Properties of some two-sample tests based on a particular measure of discrepancy. Ann. Math. Statist. **27** (1956), 1006–1016.

Let F and G be two continuous univariate distribution functions. For testing the hypothesis $F=G$ on the basis of two independent random samples of sizes m and n from F and G, respectively, E. Lehmann [same Ann. **22** (1951), 165–179; MR **12**, 726] proposed a statistic $Q_{m,n}$ which is a symmetric unbiased estimator of

$$Q(F,G)=\int (F-G)^2 d(F+G)+\frac{1}{3}.$$

Various properties of $Q_{m,n}$ and two closely related statistics are studied. In particular, the asymptotic distributions of these statistics, for $n\to\infty$, $m/n\to c$ $(0<c<\infty)$, are obtained, both when $F=G$ and (under mild restrictions) when $F\neq G$.

W. Hoeffding (Chapel Hill, N.C.).

Roy, J. On some quick decision methods in multivariate and univariate analysis. Sankhyā **17** (1956), 77–88.

The paper examines some hypotheses about the underlying distribution of some measurable characteristic using counting. The binomial distribution is utilized. Though the author does not provide general solutions for fixed sample size, he presents results for n large so he can use the normal approximation to the binomial. These tests are discussed under the headings: the most powerful binomial test of a simple hypothesis against a simple alternative; minimisation of size of sample when both kinds of error are pre-assigned; test of a simple hypothesis about a single parameter against a class of simple alternatives; asymptotically locally most powerful one sided test. The tests are illustrated by application to various examples and some comparisons are made with the corresponding most powerful classical tests.

P. S. Dwyer (Ann Arbor, Mich.).

Fisher, R. A.; and Healy, M. J. R. New tables of Behrens' test of significance. J. Roy. Statist. Soc. Ser. B. **18** (1956), 212–216.

Watson, G. S.; and Williams, E. J. On the construction of significance tests on the circle and the sphere. Biometrika **43** (1956), 344–352.

This paper considers the sampling theory for vector variates whose angular direction θ has probability density proportional to $\exp(\kappa\cos(\theta-\theta_0))$ [the distribution used by R. A. Fisher for the analysis of palaeo-magnetic data, Proc. Roy. Soc. London. Ser. A. **217** (1953), 295–305; MR **15**, 139]. The author calculates the distribution functions of various statistics, finds confidence limits for κ, and suggests statistics for the testing of a prescribed θ_0, and of the homogeneity of κ and of θ_0 over several populations. Approximations or inequalities for the significance points of the test statistics are derived.

P. Whittle (Wellington).

Romani, Jose. Distribution of the algebraic sum of Poisson variables. Trabajos Estadist. **7** (1956), 175–181. (Spanish. English summary)

Let X and Y be two independent random variables and suppose that X (Y) has a Poisson distribution with parameter θ_1 (θ_2). The author shows that the distribution of the difference $Z=X-Y$ is given by

$$(*)\qquad P(Z=n)=\exp(-\theta_1-\theta_2)\cdot(\theta_1/\theta_2)^{n/2}I_n(2\sqrt{(\theta_1\theta_2)},$$

where $I_n(t)$ is the modified Bessel function of integer order n. The particular case $\theta_1=\theta_2$ was discussed by de Castro [Portugal. Math. **11** (1952), 173–175; MR **14**, 566]. The author considers also sampling from a population with distribution (*) and shows that the sample mean is a maximum likelihood estimate for the population mean

$\theta_1-\theta_2$. Separate maximum likelihood estimates for the population parameters are however not obtained.

E. Lukacs (Washington, D.C.).

Bennett, B. M. Note on the Poisson index of dispersion. Trabajos Estadist. **7** (1956), 183–185. (Spanish summary)

Let $X_1, X_2, \cdots, X_n$ be a sample of n independent observations and assume that $X_j=U_j+V_j$ $(j=1, 2, \cdots, n)$. Suppose that (i) the U_j have the same Poisson distribution, (ii) the V_j are independent and have mean zero. The author gives formulae for the mean and the variance of the "Poisson index of dispersion" $z=ns^2/\bar{x}$. Here s^2 is the sample variance and $\bar{x}$ is the sample mean.

E. Lukacs (Washington D.C.).

Féron, R. Sur les tableaux de corrélation dont les marges sont données. Cas de l'espace a trois dimensions. Publ. Inst. Statist. Univ. Paris **5** (1956), 3–12.

M. Fréchet [Ann. Univ. Lyon. Sect. A. (3) **14** (1951), 53–77; MR **14**, 189] studied the family of bivariate distributions with given marginal distributions and proved that this family contains a maximal as well as a minimal element. The author of the present paper shows that this result can not be extended to the trivariate case. He studies first a family of trivariate distributions $F(x, y, z)$ whose one-dimensional marginal distributions $F(x, \infty, \infty)$, $F(\infty, y, \infty)$, $F(\infty, \infty, z)$ are given and shows that this family has a maximal element but no minimal element. He then considers the case where a two-dimensional marginal $F(x, y, \infty)$ and a one-dimensional marginal $F(\infty, \infty, z)$ are given, in this case the family has neither a maximal nor a minimal element.

E. Lukacs (Washington D.C.).

Watson, G. S. A note on the circular multivariate distribution. Biometrika **43** (1956), 467.

The author shows that the circular definition of the serial correlation coefficient is a good approximation when one studies seasonal variation by averaging data over many years. *R. L. Anderson* (Raleigh, N.C.).

Roy, S. N.; and Mitra, S. K. An introduction to some non-parametric generalizations of analysis of variance and multivariate analysis. Biometrika **43** (1956), 361–376.

Suppose n random variables are independent, identically distributed and subject to a two way classification. Let p_{ij} be the probability of an observation in the (i, j)-th cell. Among some of the hypotheses treated are (I) $p_{ij}=(\sum_r p_{ir})(\sum_s p_{sj})$ and (II) p_{ij} is independent of j. Similar hypotheses are studied under the assumption that the number of entries in each column and row is fixed. Similar problems for three way classification are studied. Some chi-square test criteria and results on large sample distribution are presented. A study of large sample power is promised for later publication. Applications described in the title are discussed. *M. Dwass.*

Scheffé, Henry. Alternative models for the analysis of variance. Ann. Math. Statist. **27** (1956), 251–271.

After having described the fixed effect, random, and mixed models and given some history of their early use in the nineteenth century, the author turns to getting a proper definition of interactions for the last two models. The following is proposed: the main effect of the random factor and its interactions with levels of the non-random factor form a random vector. These vectors are independent and identically distributed, but there is correlation between components of the vector. There is also some discussion of randomization models. [For further discussion and proofs the reader should refer to H. Scheffé, same Ann. **27** (1956), 23–36; MR **17**, 1102.]

S. W. Nash (Vancouver, B.C.).

Huttly, N. A. The fitting of regression curves with autocorrelated data. Biometrika **43** (1956), 468–474.

The author considers the fitting of a polynomial trend when deviations from trend follow a linear Markov process (in continuous time t) and are normally distributed. Using direct methods he calculates the variances of the least square and maximum likelihood estimates of the regression coefficients of t and t^2, and plots the efficiencies of the least square estimates as functions of sample size. *P. Whittle* (Wellington).

Mattila, Sakari. Some tests based on moving average operations on time series. Ann. Acad. Sci. Fenn. Ser. A. I. no. **226** (1956), 12 pp.

This is a sequel to an article previously reviewed [same Ann. no. **156** (1953); MR **15**, 331]. The current article presents a method of testing for the existence of a trend by use of the estimates of random variation, s_1^2, and of the trend variation, s_3^2, discussed in the 1953 article. A moving average of length k is used on a series of nk equally spaced observations; results are given here for $k=3$ and $n\geq 3$. The author obtains the variance-covariance matrix for s_1^2 and s_3^2, assuming normality, and large sample tests of the hypothesis that $\sigma_1^2=\sigma_3^2$ (no real trend component). *R. L. Anderson* (Raleigh, N.C.).

Walker, A. M. A goodness of fit test for spectral distribution functions of stationary time series with normal residuals. Biometrika **43** (1956), 257–275.

This paper is concerned with the construction of large sample tests of the hypothesis that a stationary time series has a given spectral distribution function. The process is supposed to be normal. The spectrum is not restricted to be absolutely continuous but may have discontinuities. Various test statistics are considered, some of which may also be used in the case when the spectrum is not completely specified. The power of the tests is also investigated. A numerical illustration is given.

U. Grenander (Stockholm).

Suzuki, Yukio. Note on optimal machine setting. Ann. Inst. Statist. Math., Tokyo **8** (1956), 61–64.

This note solves the problem of optimal machine setting by assuming that when a machine is set at a level μ, the measurement of an attribute of a product in which one is interested, say x, is distributed normally with mean μ and variance 1. Assuming further that all items below a certain lower limit of the attribute are to be scrapped and those above an upper limit are reworked, a simple cost function is obtained and the corresponding risk function is minimized to obtain the optimal value of μ. {The equation (4) for the solution of optimal μ can be easily derived by a simpler (and correct) method whereas the author has followed a rather difficult (and incorrect) procedure in deriving it. There is a minor misprint in equation (7) also in that a plus sign has been omitted.} Graphical solution is given for some very simple values of the constants appearing in the cost function. *Om P. Aggarwal.*

See also: Pillai, p. 456; Roy and Laha, p. 459; Ramakrishnan, p. 459; Sprott, p. 459; Gurland, p. 500; Aoyama, p. 515; Deuel, p. 516; Chaudhuri, p. 517; Saxer, p. 518; van Elteren, p. 519; Krzywoblocki, p. 532; Banerjee, p. 547; Rosner, p. 548; McHugh, p. 548; Bock, p. 548.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Mewes, E. Formeln für die Massenkräfte und kinematischen Zusammenhänge bei geschränkten Schubkurbelgetrieben. Ing.-Arch. **24** (1956), 291–298.

Es werden die Koppelwinkel und die Koordinaten eines mit der Koppel des Schub-Kurbelgetriebes starr verbundenen Punktes in eine Fourier-Reihe entwickelt mit dem Kurbelwinkel als Argument. Rekursionsformeln für die Koeffizienten. Ein numerisches Beispiel. *O. Bottema.*

Pailloux, Henri. Charges roulantes. Bull. Sci. Math. (2) **80** (1956), 46–61.

Le problème des charges roulantes sur un pont a été longuement étudié. Il a donné lieu à de nombreux développements, sans que, semble-t-il, une solution rigoureuse ait été tentée. L'auteur donne ici une mise en équation correcte de ce problème. Dans le cas d'une charge roulante montée sur ressort, la solution du problème résulte de la résolution d'une équation de Volterra de seconde espèce, possédant une solution unique; on démontre donc l'existence d'une telle solution. S'il existe différentes charges roulantes, on rencontre autant d'équations de Volterra, formant ainsi un système possédant une solution unique lorsque les charges sont montées sur ressorts. L'auteur donne encore la mise en équation du problème dans le cas d'une ou plusieurs charges roulantes adhérentes au pont. On rencontre alors une ou plusieurs équations de Volterra de première espèce dans un cas non classique, et l'existence de la solution reste a démontrer.

R. Gran Olsson (Trondheim).

Kosticyn, V. T. Some questions of friction of a flexible constraint. Trudy Inst. Mašinoved. **16** (1956), no. 62, 43–55. (Russian)

La première partie de ce Mémoire est consacrée à la généralisation de la formule d'Euler pour la friction des matériaux flexibles.

On considère le mouvement des matériaux sur la courbe avec la courbure variable et on établie la formule

$$\frac{dQ}{Q}=f\frac{ds}{\rho}$$

où Q est la force, s l'élement linéaire et ρ^{-1} la courbure de la courbe, f la friction.

On montre aussi comment on peut évaluer la friction quand l'équation de la courbe est inconnue.

Dans la seconde partie on considère le cylindre élastique et flexible qui se meut dans un tuyau et établit aussi la formule, la figure et le tableau approximatif pour la friction. Dans la troisième partie on considère la tension des matériaux élastiques qui se mouvent avec la friction entre deux plateaux; on étudie les divers cas pratiques et on détermine la validité des résultats qu'on a obtenus.

F. Vyčichlo (Praha).

Magnus, K. Stationäre Schwingungen in nichtlinearen dynamischen Systemen mit Totzeiten. Ing.-Arch. **24** (1956), 341–350.

This is a heuristic treatment of systems of nonlinear difference-differential equations (specifically, equations of mixed differences) using the method of Kryloff-Bogoliuboff. Stability is considered. *E. Pinney.*

Blaquière, Augustin. Synchronisation des oscillateurs non linéaires, critère général de stabilité. C. R. Acad. Sci. Paris **243** (1956), 2002–2004.

The author considers certain non-linear systems which have a harmonic response of the same frequency as a forcing harmonic term. He describes the application of his graphical method used earlier for linear systems [J. Phys. Radium (8) **13** (1952), 636–644; MR **14**, 800] to such non-linear systems. And he obtains some partly analytic and partly geometric criteria for stability from his graphical analysis. *P. Franklin.*

Kuhtenko, O. I. On the theory of small oscillations and the stability of motion of systems with non-holonomic constraints. Akad. Nauk Ukrain. RSR. Prikl. Meh. **1** (1955), 205–223. (Ukrainian. Russian summary)

In the vibration problem of a non-holonomic dynamical system there is a difference of opinion. E. T. Whittaker has shown [Analytical dynamics, 4th ed., Cambridge, 1937] that the vibration problem of such a system with s independent coordinates and l nonintegrable kinematical constraints reduces to the problem of a holonomic system with $s-l$ degrees of freedom since the kinematical relations of the form $\sum A_{sr}\dot{q}_s=0$ can be integrated because $A_{sr}=$const. On the other hand, O. Bottema has shown [Nederl. Akad. Wetensch., Proc. **52** (1944), 848–850; MR **11**, 470] that in such a case it is necessary to apply the Lagrange equations with undetermined multipliers. In this paper the author shows that Bottema's relations have restricted applicability, i.e. only in the case when the system vibrates about the configuration of stable equilibrium, and that in the first approximation in the case when system vibrates about the stationary state of motion the relevant zeros do not exist. One example is treated. *D. Rašković* (Belgrade).

Neumark, S. Analysis of short-period longitudinal oscillations of an aircraft-interpretation of flight tests. Aero. Res. Council, Rep. and Memo. no. **2940** (1952), 55 pp. (1956).

This paper is concerned with the mathematical problem of inferring the values of certain "aerodynamic derivatives" for aircraft from flight tests of how particular variables respond to an elevator input of square shape. Curves of rate of pitch, normal acceleration factor and elevator displacement are corrected by a filtering process to eliminate the long-period "phugoid" mode. The frequency, damping factor, amplitude ratios and phase differences in the remaining short-period oscillations are then used to deduce the aircraft lift slope, the damping in pitch, the manoeuvre margin and other relevant variables. The analysis is given for the cases when the elevator is either fixed or free after the initial square imput, and when the aircraft has a conventional tail (stabilizer) or is tailless.

M. J. Lighthill.

Pearcey, T.; and Hill, G. W. The accelerated motion of droplets and bubbles. Austral. J. Phys. **9** (1956), 19–30.

The authors study the effect of past history on the equation of motion of small spherical objects, such as droplets and bubles, moving through a viscous medium. Under certain simplifying assumptions, and by reduction to non-dimensional form, the equation obtained is

$$\frac{dV}{d\tau}+V+\lambda\int_0^{\infty}\frac{dV(\tau-\tau')}{d(\tau-\tau')}\frac{d\tau'}{\sqrt{\tau'}}=f,$$

in which V is the velocity, τ the time, f the applied body force and λ a parameter. To illustrate the effect of past history upon the velocity and position of the object, the authors take f equal to unity for $\tau<0$ and zero for $\tau>0$. In the absence of any effects of past acceleration ($\lambda=0$) the velocity for $\tau>0$ would be $e^{-\tau}$. The difference, w, in the velocity due to the effect of past history satisfies the integro-differential equation

$$\frac{dw}{d\tau}+w+\lambda\int_0^{\tau}\frac{dw(\tau-\tau')}{d(\tau-\tau')}\frac{d\tau'}{\sqrt{\tau'}}=\lambda e^{-\tau}\int_0^{\tau}e^{\tau'}\frac{d\tau'}{\sqrt{\tau'}}\quad(\tau>0).$$

Starting with $w=0$ at $\tau=0$, the solution of this equation is obtained with the help of the CSIRO digital computer for λ in the range $0.1\leqq\lambda\leqq10.0$. *C. J. Bouwkamp.*

Hitchcock, H. P. Fundamental interior ballistic equations. Ordnance Computer Research Report, Ballistic Research Laboratories, Aberdeen Proving Ground, Md. vol. 3 (1956), no. 4, pp. 6–10. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D.C.)

A brief report explaining the equations upon which Bennett's "Tables for interior ballistics" [Ordnance Dept. U. S. Army, 1921] were based. The occasion is the issuance of an extensive new edition computed by electronic digital methods under the direction of the author. Only minor modifications in form other than notable enlargement of range have been deemed necessary. The equations were based upon those viewed as most satisfactory in 1920 by Col. W. H. Tschappat (later General, and Chief of Ordnance) and were at that time reduced to essential dimensionless variables by Bennett. Experiments in the last thirty-five years have rendered obsolete two auxiliary formulas for computing certain parameters analytically, mentioned in the original explanation. *A. A. Bennett* (Providence, R.I.).

Blitzer, Leon; and Wheelon, Albert D. Maximum range of a projectile in vacuum on a spherical earth. Amer. J. Phys. **25** (1957), 21–24.

Fried, Burton D.; and Richardson, John M. Optimum rocket trajectories. J. Appl. Phys. **27** (1956), 955–961.

The introductory remarks of the authors state very clearly the results of this article. Some general considerations concerning maximum range missile trajectories are presented. If the mass M and the thrust magnitude F are given functions of the time t, if the rocket is treated as a point mass, and if aerodynamic effects are neglected, then on the assumption that the external force field is constant, the maximum range trajectory is one for which the thrust direction is fixed in space throughout the powered flight. This result is independent of the nature of the functions $M(t)$ and $F(t)$, although the value of the thrust attitude angle ψ depends on these functions. If the constant force field is replaced by a central inverse square field, the maximum range is no longer given by a constant thrust attitude trajectory. However, for most cases of general interest, the optimum attitude is very nearly constant. Finally there is a discussion of the relation of these results to the classical problems of maximum range for a projectile. The details of the proofs of these theorems are concisely carried out. *J. De Cicco.*

Kooy, J. M. J. On the calculation of the powered flight of a long range rocket, supervised by an automatic pilot. Astronaut. Acta **1**, (1955), fasc. 4, 191–198.

The paper describes a method for computing the flight of a long-range rocket, taking into account thrust, aerodynamic forces, automatic control, gravity and Coriolis force. *M. J. Lighthill* (Manchester).

See also: Grabar', p. 483; Reissig, p. 484; Yuščenko, p. 527; Blitzer, p. 544.

Statistical Mechanics

Lessen, Martin. Thermoelasticity and thermal shock. J. Mech. Phys. Solids **5** (1956), 57–61.

The author derives thermoelastic stress-strain equations and shows additional term is required in heat conductivity equation [cf. alternative derivation by M. A. Biot, J. Appl. Phys. **27** (1956), 240–253; MR **17**, 1035]. The thermal shock problem is solved assuming dynamic terms can be neglected. *D. R. Bland* (London).

Prokof'ev, V. A. On taking account of radiation in one-dimensional steady motion of a monatomic gas. Moskov. Gos. Univ. Uč. Zap. **172** (1954). Meh. 5, 79–124. (Russian)

Glauberman, A. E. On the theory of dipole crystals in the external electric field. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 49–52. (Russian)

Karle, J.; and Hauptman, H. A theory of phase determination for the four types of non-centrosymmetric space groups $1P222$, $2P22$, $3P_12$, $3P_22$. Acta Cryst. **9** (1956), 635–651.

A continuation of previous work on centrosymmetric structures [Hauptmann and Karle, Acta Cryst. **6** (1953), 131–135, 136–141; **8**, 355; **9** (1956), 45–55; Solution of the phase problem, I, Amer. Cryst. Assoc., Wilmington, Delaware, 1953; MR **17**, 583; **16**, 779]. General expressions are derived for the probability that the real and imaginary parts of two or more structure factors have specified values simultaneously. Useful special cases are evaluated for certain space groups in which some structure factors are purely real, i.e. whose phases can only be 0 or π. The useful cases are those in which one structure factor is related to the modulus of one or more others. For example, in space group $P222$, the joint probability distribution of unitary structure factors E_1, E_2 whose indices satisfy the relations $h_1=2h_2\neq0$, $k_1=2k_2\neq0$, $l_1=0$, $l_2\neq0$, is

$$P(E_1,|E_2|)=2(2\pi)^{-1}|E_2|\exp\{-\tfrac{1}{2}E_1^2-|E_2|^2\}\times[1+S_3S_2^{-3/2}E_1(|E_2|^2-1)],$$

where S_n is the sum of the nth powers of the atomic numbers of the atoms in the unit cell. From this it is easy to find a relation between E_1 and the average value of $|E_2|^2-1$ which implies that

$$E_{2h,2k,0} \approx S_2^{3/2} S_3^{-1} \langle |E_{hkl}|^2-1 \rangle_l,$$

where $\approx$ means 'is probably equal to' and the averaging is over all values of l. Precisely expressed, E_2 is distributed normally about the value stated with a variance of $S_2S_4S_3^{-2}-1$, which is small if there is not much difference between the atomic numbers of the atoms in the cell. If E_2 is not too small this relation suffices to establish its phase, even if the numerical agreement is not close.

Relations similar in principle but more complex in form give the phase angles for structure factors with no zero index ($hkl \neq 0$).

The space groups $P222$, $P2_12_12_1$ and $P422$ are treated in detail; the computations for certain others are stated to be routine but tedious. *A. J. C. Wilson* (Cardiff).

Trlifaj, Miroslav. Thermal dissociation energy and the basic state of F_2^+ centres. Czechoslovak J. Phys. **6** (1956), 99–107. (Russian. English summary)

Phillips, James C. Critical points and lattice vibration spectra. Phys. Rev. (2) **104** (1956), 1263–1277.

This paper extends the classification of singularities in the lattice vibration spectra of crystalline solids previously described by Van Hove [Phys. Rev. (2) **89** (1953), 1189–1193; MR **15**, 88] to include the singularities that arise from multiple roots of the secular equation at critical wave vectors. It is shown from group theoretical arguments how critical wave vectors must exist at certain symmetry points of the wave vector space, and that the frequencies are likely to be non-analytic functions of the wave numbers at these points. The types of critical points, particularly those at the non-analytic points, are then classified and the minimum number of such points determined. Several specific examples are considered in which the actual number of critical points is equal to this minimum number. A scheme for finding an approximate frequency distribution is proposed using the knowledge of the critical points and applied to the spectrum of aluminum. *G. F. Newell* (Providence, R.I.).

See also: Betts, Bhatia and Wyman, p. 475; Seeger, p. 489; Welander, p. 528; Zeludev, p. 535; Jansen and Solem, p. 535; Callen, p. 537.

Elasticity, Visco-elasticity, Plasticity

Desoyer, K. Zur rollenden Reibung zwischen Scheiben mit verschiedenen Elastizitätskonstanten. Z. Angew. Math. Mech. **36** (1956), 274–276.

Für gleiche Elastizitätskonstanten ist das Problem gelöst [Föppl, Die strenge Lösung für die rollende Reibung, Leibniz-Verlag, München, 1947; Heinrich, Österreich. Ing.-Arch. **4** (1950), 363–375]. Verf. gibt die Gleichungen an und löst sie für die Fälle: gleiten bzw. haften im ganzen Berührgebiet. Im allgemeinen gibt es ein hinteres und ein vorderes Gleitgebiet und ein Haftgebiet.

O. Bottema (Delft).

Jobert, Georges. Déformation plane d'un solide élastique isotrope et hétérogène. C. R. Acad. Sci. Paris **244** (1957), 555–558.

Pickett, Gerald; and Sundara Raja Iyengar, K. T. Stress concentration in post-tensioned prestressed concrete beams. J. Tech. Bengal Engrg. Coll. **1** (1956), 105–112.

Wegner, Udo. Berechnung von teilweise eingespannten rechteckigen Platten bei Vorgabe von Randmomenten. Z. Angew. Math. Mech. **36** (1956), 340–355. (English, French and Russian summaries)

Die Analyse der mechanischen und geometrischen Grössen bei teilweise eingespannten rechteckigen Platten bereitet erstens dadurch Schwierigkeiten, dass man durch die notwendigen Überlagerungen von Gleichungen der Biegefläche zur Befriedigung der Einspannbedingung gezwungen wird, lineare Gleichungen mit unendlich vielen Unbekannten aufzulösen, deren Lösungen in alle Beiwerte der überlagerten Gleichung eingehen. Zweitens weiss man nicht, nach welchem System von Funktionen die genaue Lösung der Bipotentialgleichung bei den oben genannten Randwertaufgaben entwickelt werden soll, um derartige unendliche Prozesse zu unterbinden.

In dieser Arbeit wird daher ein nicht-vollständiges, orthogonales System von biharmonischen Funktionen untersucht, das bei vielen der im Titel der Arbeit genannten Aufgaben die Beiwerte in geschlossener Form darzustellen erlaubt, ähnlich der Darstellung nach Fourier. Das Verfahren wird an einigen technisch wichtigen Problemen erläutert.

Zur Darstellung wird ein komplexwertiges biharmonisches System von Funktionen der reellen Variablen benutzt, das von J. Fadle [Ing.-Arch. **11** (1940), 125–149; MR **2**, 30] zur Bestimmung von Eigenwertfunktionen des ebenen Selbstspannungszustandes der quadratischen Scheibe sowie von G. Worch [Forschungsh. Gebiete Stahlbaues **6** (1943), 189–199] für die angenäherte Darstellung der Biegefläche von allseitig eingespannten Rechteckplatten mit Erfolg benutzt wurde.

R. Gran Olsson (Trondheim).

Nikol'skiĭ, E. N. Strains and stresses in cylindrical shells and thin walled rods with non-flexible contour of cross-section. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 6, 45–54. (Russian)

The author analyzes a cylindrical shell of arbitrary shape under the assumption that the shape of transverse crossections of the shell does not change when the shell deforms under the action of applied loads. An idealized model of the shell is used for the purpose of analysis. The shell is assumed to be made up of a membrane stretched over transverse ribs spaced infinitely close together. These ribs are perfectly rigid and can perform only rigid body movements with respect to each other. The membrane is perfectly flexible and capable of withstanding normal and shearing forces in its plane only. The attachment of the membrane to the ribs is such that only forces normal to the surface of the membrane are transmitted from the ribs to the membrane and vice versa, and that displacements of points of the membrane with respect to the ribs are only possible in directions tangential to the surface of the membrane.

The active external loads are (a) surface loads and moments acting on the ribs, and (b) in the case of an open shell, also edge loads acting on the straight edges of the membrane. The reactive forces at the supports of the shell are assumed to act on the two curved edges of the membrane.

One equation of compatibility for the membrane and

the ribs, and three equations of force equilibrium for an infinitesimal membrane element are derived. Three additional equations are obtained by considering the equilibrium of a finite portion of the membrane.

For the stresses a function of the Airy type is assumed, thus satisfying two conditions of equilibrium of the membrane element. The external loads are developed into Fourier series, and the solutions for the remaining five equations are assumed in form of Fourier series. Some of the Fourier coefficients here must be found as solutions of linear differential equations.

The author points out that, for a shell with non-deformable crossections subjected to edge loads only, the general solution can be represented as a superposition of two solutions. The first is a plane stress solution obtained by treating the curved membrane as a plane surface which is acted upon by the given edge loads. The second solution is a membrane solution for the given cylindrical shell with non-deformable crossections acted upon by surface loads which are equal and opposite to those found in the solution (1) to be necessary to maintain the unsupported membrane in its curved shape.

An experiment is described, and it is shown that measured stresses agree well with those predicted by the theory. *H. V. Hahne* (Palo Alto, Calif.).

Kačanov, L. M. Creeping in oval pipes and pipes with variable wall thickness. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1956, no. 9, 65–71. (Russian)

Deformations and stresses due to creep are considered for plane flow in straight tubes at uniform temperature and with uniform over-pressure p acting on the interior surface. The tubes are considered slightly imperfect in that boundary curves of the cross section are not quite concentric circles; deviations from perfection are measured by a dimensionless, small parameter λ. Three imperfect geometries treated are: a) two nearly circular but non-concentric curves; b) two concentric and similar (nearly constant wall thickness) ovals; c) two concentric but dissimilar ovals. The creep law used is that of non-Newtonian viscosity:

$$e_{ij}=BT^{m-1}s_{ij}\ (m\geqq 1),$$

where e_{ij} is the strain rate tensor, s_{ij} the stress deviation tensor, and T the square root of the second stress invariant. The problem is linearized by a perturbation procedure in λ, and results are presented for the first order perturbations. *W. Nachbar* (Palo Alto, Calif.).

Kopzon, G. I. Vibration of a slightly sloping wing-shell in a gas stream. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 377–380. (Russian)

The author considers an eighth order partial differential equation for vibration of a wing subject to linearized supersonic aerodynamics. If the displacements are expressed as $q=\sum_0^\infty q_n(y)\exp(i\omega_n t+\alpha_n x)$, where $\omega_n=2\pi n/\tau$, $\alpha_n=2\pi n/L$ and x (y) is a span (chord)-wise coordinate, then $q_n(y)$ satisfies an integro-differential equation of convolution type. As in an earlier note [same Dokl. (N.S.) **107** (1956), 217–220; MR **18**, 163] the author solves this by means of Laplace transforms. The convergence of the series for q is discussed; lift, drag, and moment coefficients due to the vibrations are found; and a calculation of the characteristic frequencies by a modification of the Rayleigh-Ritz method is outlined. *J. Giese.*

Volterra, E. Eigenvibrations of curved elastic bars according to the "method of internal constraints". Ing.-Arch. **24** (1956), 317–329.

In the present paper it is shown how it is possible to discuss the problem of vibrations of curved bars once the solution of the corresponding problem for the straight bar is known. The method of approach in deriving the equation of motion is completely contained in the constraint equation of the problem and in the application of the principle of Hamilton, while the method of solving the equations is based on a perturbation method which utilizes as first approximation the solution of the straight case.

More general dynamic problems of elongated solids in which the inertia characteristics of the cross-section of the bar are known functions of the independent variable can be attacked and solved by the same method which has so far been demonstrated for the simple cases of bars having constant inertia characteristics. [See also Volterra, Ing.-Arch. **23** (1955), 402–409; MR **17**, 556.]

R. Gran Olsson (Trondheim).

Onoe, Morio. Contour vibrations of isotropic circular plates. J. Acoust. Soc. Amer. **28** (1956), 1158–1162.

Using a theory developed by A. E. H. Love [Mathematical theory of elasticity, Cambridge, 4th ed., 1927, p. 497] for contour vibrations involving no transverse displacements, the author shows that: (a) Coefficients of dilatation and rotation are equal regardless of Poisson's ratio only for modes of circumferential order one. (b) The first mode of order two has lowest natural frequency. This mode is practically equivoluminal, while the second mode is dilatational. (c) Circular nodal lines exist only for radial and tangential modes. Frequencies and displacements for the more important modes are calculated.

For barium titanate vibrators, it is shown how the radial and compound modes can be excited while the tangential modes can never be excited. Excitation of compound modes provides a convenient measurement of Poisson's ratio.

For thick circular plates, the vibrations of certain compound modes which are almost equivoluminal can be approximated by rotation only. Comparisons with the thin plate results reveals that the accuracy of approximation is satisfactory. *H. D. Conway* (Ithaca, N.Y.).

Klein, Bertram. Vibration of simply supported flat plates simultaneously tapered in planform and thickness. J. Acoust. Soc. Amer. **28** (1956), 1177–1181.

Solutions are given for the first three natural frequencies of symmetrical flexural vibration, using a collocation method in which the differential equation is satisfied at a selected number of points. Results are presented graphically and cover most practical tapers. *H. D. Conway.*

Gold, Louis. Rayleigh wave propagation on anisotropic (cubic) media. Phys. Rev. (2) **104** (1956), 1532–1536.

The essential features of surface-wave propagation for elastic waves on anisotropic media are delineated by consideration of the solution for cubic crystals. In using the coordinate system defined by the surface and the direction normal to it, transformation of the elastic coefficient tensor is required. Conventional means for doing this can be prohibitively laborious; but by invoking the isotropy condition, the calculation becomes quite amenable. *From the author's summary.*

Kaul, R. K.; and Cadambe, V. The natural frequencies of thin skew plates. Aero. Quart. **7** (1956), 337–352.

This paper deals with the frequencies of rhombus-shaped plates. The Rayleigh-Ritz method is briefly explained and then wrongly applied. The authors appear to believe that if the Rayleigh quotient is formed with each of k orthogonal functions and the numbers so obtained written in increasing order, then they are upper bounds for the first k eigenvalues. That this is wrong can be seen by forming the Rayleigh quotients with (u_1-u_2) and (u_1+u_2), where u_1 and u_2 are the first two eigenfunctions. The two quotients so obtained are equal and lie between the first and second eigenvalues.

Lower bounds are found by the method of Kato [J. Phys. Soc. Japan **4** (1949), 334–339; MR **12**, 447] and Temple [Proc. Roy. Soc. London. Ser. A. **211** (1952), 204–224; MR **13**, 691]. However, it is first incorrectly stated that the method of A. Weinstein [Etude des spectres des équations aux dérivées partielles de la théorie des plaques élastiques, Gauthier-Villars, Paris, 1937] does not give lower bounds for the higher eigenvalues.

H. F. Weinberger (College Park, Md.).

Šemyakin, E. I. Propagation of unsteady disturbances in a visco-elastic medium. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 34–37. (Russian)

A method is proposed for the solution of certain boundary problems dealing with propagation of unsteady disturbances in a visco-elastic medium. First a solution as a double (Fourier-Mellin) integral is obtained by the method of incomplete separation of variables. Then a transformation of Efros is applied. The final solution as a function of coordinates and the time t is represented as an integral in which the integrand is the product one of whose factors is the solution corresponding to the elastic problem for the δ-function, and the other is a certain additional function $R(t,\tau)$. This last function is determined as the solution of the simple differential equation

$$\left(1+\omega\frac{\partial}{\partial t}\right)\frac{\partial^2 R(t,\tau)}{\partial t^2}=\frac{\partial^2 R(t,\tau)}{\partial t^2}$$

satisfying the conditions

$$R(t,\tau)\Big|_{t=0}=\frac{\partial R(t,\tau)}{\partial t}\Big|_{t=0}=0 \text{ and } \left(1+\omega\frac{\partial}{\partial t}\right)R(t,\tau)\Big|_{\tau=0}=a(t),$$

where $\omega=$const, and $a(t)$ is the source function.

An example is given of the solution of the simplest problem in the case of halfspace.

K. I. Ogurcov (RŽMeh 1956, no. 7790).

Czitary, E. Verhalten eines Drahtseiles auf einer Rolle mit elastischem Futter. Österreich. Ing.-Arch. **10** (1956), 349–359.

Uflyand, Ya. S. Solution of a spatial problem of the theory of elasticity for a wedge-shaped body with given displacements on the boundary. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 1177–1179. (Russian)

A formal solution is presented for the elastic deformation of an infinite wedge with given displacements on the faces. It is first shown how, in a cylindrical coordinate system, the displacements, or suitable auxiliary functions defined from them, can be given a double integral representation in the form of a Fourier integral in Z and a Kontorovich-Lebedev integral in r. Each of the three (θ-dependent) coefficients contains two arbitrary functions of the variables of integration. If the displacements on the faces are sufficiently well-behaved they, or their corresponding auxiliary functions, can be given similar representations with known coefficients. The solution is then completed by equating coefficients and solving for the six unknowns. Four of these can be found by algebraic manipulation. To find the remaining two it is necessary to solve a pair of integral equations. This is done with the aid of Fourier transforms.

P. Mann-Nachbar (Palo Alto, Calif.).

Yuščenko, A. A. On the equations of vibration of an elastic-viscous thread with a variable mass on one end. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 235–237. (Ukrainian. Russian summary)

The author assumes that the mass attached at the end of a hoisting thread varies according to a linear law with respect to the time t. The solution of the original third order partial differential equation for $u(x,t)$, the absolute lengthening of a segment of the thread of length x, corresponding to zero initial values of $u(x,t)$ and $u_t(x,t)$ and given boundary conditions at the ends of the thread, is reduced, using an averaging method of Yu. D. Sokolov [Dopovidi Akad. Nauk Ukrain. RSR **1955**, 21–25; MR **17**, 307], to a system of two ordinary differential equations of the second order. *E. Leimanis* (Vancouver, B.C.).

Finzi, Leo. Modelli di corpi plastici tratti da una analogia con strutture a guscio. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 1000–1019.

The author attempts to establish a membrane analogy for the plane-strain problems of the theory of plasticity. The starting point of the proposed analogy is the following well-known result [see, e.g., A. E. Green and W. Zerna, Theoretical elasticity, Oxford, 1954, p. 410; MR **16**, 306]: If the general equations of equilibrium of the membrane theory of shells are referred to a base plane, then two of these equations may be reduced to the equations of equilibrium of the plane-strain problem by introducing appropriate weighted stress resultants. The author's contention is that the third equation of equilibrium may then play the role of the yield condition of plasticity if the form of the membrane shell and the loads acting on it are chosen properly. In other words a loaded membrane shell could be associated with a fully plastic stress field. This viewpoint enables the author to develop a semi-experimental iterative method for the construction of fully plastic stress fields.

{It must be noted that this method is applicable only to the fully plastic problems and the method of characteristics is successfully employed for the solution of these problems.} *E. T. Onat* (Providence, R.I.).

Hodge, P. G., Jr. Minimum principles of piecewise linear isotropic plasticity. J. Rational Mech. Anal. **5** (1956), 917–938.

This paper deals with a theory of plasticity in which the yield function is a piecewise linear function of the stresses, the flow law is based on the plastic potential, and the strain-hardening is isotropic and linear. First the flow law at a "corner" of an isotropic yield function is derived in full generality. Next it is shown that the flow law in the piecewise-linear case can be integrated in such a way that some of the advantages of a deformation theory are attained. Finally, minimum principles which are analogous to the minimum complementary and potential energy theorems of elasticity, but which are valid only under certain rather restrictive conditions, are established.

The author comments that, in spite of the severity of the limitations, these principles already have been found to be useful in specific applications to plastic shells.

F. B. Hildebrand (Cambridge, Mass.).

Grandori, Giuseppe. Travi elastoplastiche soggette a carichi di intensità comunque variabile fra limiti assegnati. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 244–254.

The author first reviews the Bleich-Melan theory of shakedown in a truss with elastic, perfectly plastic members. He then points out that there are certain difficulties in applying this theory to beams of solid cross section, because the relation between bending moment and curvature approaches the fully plastic behavior only in an asymptotic manner. *W. Prager.*

See also: Borg, p. 474; Phillips, p. 525.

Fluid Mechanics, Acoustics

Oswatitsch, Klaus. Über eine Verallgemeinerung des Potentiales auf Strömungen mit Drehung. Österreich. Ing.-Arch. **10** (1956), 239–241.

Definiert man das Strömungspotential ϕ dadurch, daß von den Flächen ϕ=const. Orthogonalität zur Geschwindigkeitsrichtung $\mathfrak{w}$ verlangt wird, ist also

$$\text{grad } \phi = f(x, y, z)\mathfrak{w} \quad (f(x, y, z) \text{ Skalarfunction}),$$

so ist ϕ auch bei Wirbelströmungen sinnvoll erklärt. Die Maßfunktion f, für die w^2 grad $f = -f\mathfrak{w} \times \text{rot } \mathfrak{w}$ gilt, wird dann mit einigen kinematischen und physikalischen Strömungseigenschaften verknüpft. *K. Maruhn.*

Armellini, Giuseppe. Sopra un limite inferiore della densità massima di una massa gassosa ruotante. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 102–107.

Vorgelegt sei eine im relativen Gleichgewicht befindliche gasförmige, wie ein starrer Körper rotierende Masse, bei der der Druck p und die Dichte ρ durch $p = k\rho^\gamma$ verknüpft sind. Unter der Annahme, daß die Flächen konstanten Druckes rotationssymmetrisch (nicht notwendig einfach zusammenhängend) und konzentrisch geschichtet sind, wird gezeigt, daß im Inneren der Masse ein Punkt Q existiert, in dem die Dichte der Ungleichung

$$\rho(Q) > \left(\frac{\gamma}{\gamma - 1}\right) \frac{\omega^2}{2\pi f}$$

genügt, (ω Winkelgeschwindigkeit, f Gravitationskonstante). Der Satz gilt auch, wie sich zeigt, für spezielle Klassen gasförmiger Körper ohne Rotationssymmetrie.

K. Maruhn (Dresden).

Quilghini, Demore. Il potenziale gravitazionale di una massa di gas perfetto in equilibrio e in condizioni isotermiche. Boll. Un. Mat. Ital. (3) **11** (1956), 422–426.

Dafür, daß eine ruhende gasförmige Masse unter der Annahme $p = k\rho$ (p Druck, ρ Dichte, k constant) im statischen Gleichgewicht eine kugelsymmetrische Gestalt besitzt, ist charakteristisch, daß das Gravitationspotential $V(r)$ der Differentialgleichung $V'' + 2r^{-1}V' = -\lambda^2 e^{-V + eV(r)}$ mit den Anfangsbedingungen $V(0) = V_0$, $V'(0) = 0$ genügt (λ konstant). Die Lösung wird mittels sukzessiver Approximationen hergestellt. *K. Maruhn* (Dresden).

Welander, Pierre. Studies on the general development of motion in a two-dimensional, ideal fluid. Tellus **7** (1955), 141–156.

This is a pleasantly discursive discussion of general questions relevant to turbulence of a strictly two-dimensional character, a concept that may have value for treatments of the motion of stably stratified fluid (particularly in the presence of Coriolis force). The most striking point is the contrast with the properties of the more familiar three-dimensional turbulence, in which energy passes continually down the scale of eddy sizes to eddies small enough for viscosity to dissipate it. The final state in two-dimensional turbulence is a single enormous vortex, and the author shows how the different stages of the approach to this equilibrium state probably take place with increasing rapidity, in such a way that only a finite time is needed to attain final equilibrium. The results of experiments with a bearing on this general result are shown. There is also much discussion of the light thrown by classical-mechanics theory on the analysis of the approach towards equilibrium. The point-vortex model of Onsager [Nuovo Cimento (9) **6** (1949), 279–287; MR **12**, 60] may have value in this analysis. *M. J. Lighthill.*

Stoker, J. J. On radiation conditions. Comm. Pure Appl. Math. **9** (1956), 577–595.

Deux travaux récents de l'Auteur étaient consacrés à l'étude à l'infini des solutions de deux problèmes particuliers de la théorie des ondes: celui des ondes de gravité en profondeur infinite [Proc. Symposia Appl. Math., v. 5, McGraw-Hill, New York, 1954, pp. 97–102; MR **15**, 997] et celui de la propagation d'une perturbation dans un écoulement uniforme dans un canal découvert [Comm. Pure Appl. Math. **6** (1953), 471–481; MR **15**, 660].

Les deux premières parties du présent travail reproduisent, en laissant de côté certains développements, le contenu des deux publications ci-dessus.

La troisième partie est consacré à l'étude à l'infini, dans un domaine non borné du plan, de l'équation $\Delta^a f + f = 0$ qui intervient dans la théorie des petites vibrations synchrones d'un milieu, ainsi que dans certains problèmes d'ondes de gravité.

Un théorème d'unicité est donné, qui montre le rôle joué par la condition à l'infini de Sommerfeld, et qui permet à l'Auteur de retrouver par une voie très directe et élégante la solution du problème du comportement d'une onde plane en présence d'un obstacle constitué par une demi-droite. *R. Gerber* (Toulon).

Biesel, F.; et Le Méhauté, B. Mouvements de résonance à deux dimensions dans une enceinte sous l'action d'ondes incidentes. Houille Blanche **11** (1956), 348–374, discussion 346.

Serre, F. Contribution à l'étude des ondes longues irrotationnelles. Houille Blanche **11** (1956), 375–390, discussion 346.

Hall, I. M. The displacement effect of a sphere in a two-dimensional shear flow. J. Fluid Mech. **1** (1956), 142–162.

This paper is an attempt to explain the observation by Young and Maas [Aero. Res. Comm., Rep. and Memo. no. **1770** (1936)] that a pitot tube in a shear flow registers a pressure equal to the stagnation pressure on a streamline displaced in the direction of higher velocities from that which approaches the tube directly along its axis. It

is shown that two-dimensional theories are inadequate to account for the magnitude of the observed effect. By considering a "spherical pitot tube", however, in a three-dimensional theory, which takes into account the stretching for vortex lines in the neighbourhood of the stagnation point of the sphere, an effect much more closely of the right order of magnitude is brought to light.

To simplify the calculations, it is assumed that the rate of stretching of fluid elements at right angles to the plane of symmetry $z=0$ of the flow is to a good approximation the same as in the irrotational flow past the sphere. The velocity field in the plane of symmetry can then be calculated independently of the rest of the velocity field by expansion in powers of the shear parameter $K=Aa/U$ (where the oncoming velocity is $U+Ay$ and the sphere has radius a and centre at the origin). Care has to be taken with the boundary condition at infinity. Finally the upstream displacement δ of the stagnation streamline is found in the form

$$\frac{\delta}{a}=1.24K-1.18K^3+O(K^5).$$

M. J. Lighthill (Manchester).

★**Robinson, A.; and Laurmann, J. A. Wing theory.** Cambridge, at the University Press, 1956. ix+569 pp. $13.50.

This is an admirable compendium of the mathematical theories of the aerodynamics of aerofoils and wings. Almost all the important results are referred to, even though there can be only a brief reference to literature in connection with the more difficult topics. It begins with a remarkably economical account of the fundamentals of hydrodynamics, including all the principal results concerning irrotational and rotational inviscid flow, the elements of viscous flow in so far as they are needed for a simple account of boundary layer theory, and some account of turbulent flow and wakes. The second chapter gives the two-dimensional aerofoil theory for incompressible flow. After the standard complex variable theory, the different methods of calculating pressure distribution for arbitrary aerofoils, or designing aerofoils for given pressure distribution, are set out. The limitations of the Kutta-Joukowski condition, and the preferability of using empirical values of lift to fix circulation, are clearly stated. In addition, biplanes and cascades of aerofoils are treated, and the chapter ends with an account of the Squire & Young method of calculating profile drag.

Chapter 3 is concerned with three-dimensional wing theory for incompressible flow. The fundamentals of thin-wing theory, including lifting-line theory and the many approaches to lifting-surface theory, are clearly described, and followed by a good account of the mathematics of wind-tunnel interference.

Chapter 4 is a very long chapter on compressible flow theory, over three-quarters of which is on linearized theory. There are some peculiar points of order here, long stretches of the difficult Hadamard theory for three-dimensional flows being put before the simple two-dimensional theory, but the final impression is of a well-balanced account.

In the non-linear sections, two-dimensional supersonic aerofoil theory might usefully have been given a few more pages, with details of the force and moment results from Busemann's quadratic theory and more details of "shock-expansion" theory; study of these points gives one a more realistic view of the probable value of three-dimensional linear theories. In the transonic-flow section, experimental data would be a useful addition.

The last chapter is a comprehensive one on theories of unsteady flow about aerofoils, both for incompressible and compressible flow. The book ends with two indexes, both thorough in scope. It is good-looking and pleasant to handle. It should be an invaluable introduction to wing aerodynamics for mathematically-minded students, as well as a solid stand-by for purposes of reference for all workers in this and allied fields. *M. J. Lighthill.*

Spence, D. A. The lift coefficient of a thin, jet-flapped wing. Proc. Roy. Soc. London. Ser. A. **238** (1956), 46–68.

A two-dimensional airfoil in plane, steady, inviscid, incompressible flow has a thin jet issuing from its trailing edge at a small angle. It is shown that the effect of the jet on the flow field is the same as that of a vortex sheet having the same shape, together with a distribution of doublets. The strength of the vortex sheet is proportional to its curvature and the jet momentum; the doublet strength is proportional to the jet thickness. In the present case the latter is ignored, and the velocity field of the vortex sheet and airfoil is calculated. When the condition is set down that the jet must lie along a streamline, a pair of simultaneous integrodifferential equations is obtained for the airfoil vortex distribution and the slope of the jet. It is next assumed that the jet lies nearly in the stream direction; the integrals can then be simplified. A solution is constructed for the special case of a flat airfoil, numerical values for which are tabulated here for 13 values of the jet-momentum coefficient. Numerical results are given for lift, chordwise lift distribution, pitching moment, and jet shape. Comparison is made with tests carried out in a wind tunnel at the National Gas Turbine Establishment. Finally, it is pointed out how the theory might be improved by iteration.

[For related studies of the same problem, see Helmbold, J. Aero. Sci. **22** (1955), 341–342; Legendre C. R. Acad. Sci. Paris **242** (1956), 2438–2440; MR **17**, 1251; Malavard, ibid. **242** (1956), 2440–2442; MR **17**, 1251; Siestrunck and Bernard, ibid. **242**, (1956) 2443–2445; MR **17**, 1251. The present author also gave a brief summary of his work in J. Aero. Sci. **23** (1956), 92–94.] *W. R. Sears.*

Woods, L. C. On the theory of source-flow from aerofoils. Quart. J. Mech. Appl. Math. **9** (1956), 441–456.

Le fluide est incompressible, l'écoulement stationnaire. L'auteur étudie l'influence sur l'écoulement d'une source placée au fond d'un conduit découpé sur le profil. Il s'agit donc d'un problème en quelque sorte intermédiaire entre l'étude classique du cas où la source est placée directement sur le profil, et celle de l'effet de jet.

La première partie est consacré à l'étude rigoureuse du problème. La solution est obtenue au moyen d'ingénieuses représentations conformes. Les efforts sont calculés à l'aide de formules de Blasius généralisées. L'auteur fait ensuite des applications au cas des profils minces: les calculs peuvent être alors effectués complètement. Il envisage le cas où la source est simplement placée sur la surface, puis celui d'une source orientée placée au bord de fuite. Une formule simple permet alors d'exprimer la portance en fonction de l'incidence, et du débit et de l'angle de la source orientée. *P. Germain* (Lille).

Edmunds, D. E. The moving aerofoil in the neighbourhood of a plane boundary. Quart. J. Mech. Appl. Math. **9** (1956), 400–424.

The author derives expressions for the lift and moment forces acting on a general, two-dimensional aerofoil which is in unsteady motion close to a plane boundary. The fluid medium is taken to be inviscid and incompressible. The method employed is an extension of that first used by Green [Quart. J. Math. Oxford Ser. **18** (1947), 167–177; MR **9**, 113] and the effects of circulation are included but trailing vortices are not. General results are first obtained and then three special aero-foils are treated in detail. These are, the flat plate aerofoil, a symmetrical aerofoil of small thickness and a circular arc aerofoil of small camber. One of the practical problems included in the paper is that of an aircraft taking off or landing. *G. N. Lance* (Southampton).

Legendre, Robert. Formation d'un remous au bord d'attaque d'un profil d'aile. C. R. Acad. Sci. Paris **243** (1956), 838–840.

The author considers inviscid flow around a wing profile for the case of a sizeable angle of incidence with the free-stream flow, as a limit of viscous flow when the viscosity goes to zero. The classical Joukowski limit pattern stands in contradication to the physical observation that, near the leading edge, there is a region of vorticity which is mainly independent of viscosity. The author constructs an inviscid flow pattern which allows for this region of vorticity. The method involves superposition and conformal mapping to set up a complex potential function. The boundary B of the region R is then defined by the requirement that the flow along B be tangential; this leads to an involved integral equation. Rather than pursue this, the author determines the boundary B by a geometrical method of successive approximations. *R. B. Davis* (Syracuse, N.Y.).

★ **v. Schlippe, B. Strömung von Flüssigkeiten mit temperaturabhängiger Zähigkeit (Kühlung von Ölen).** Forschungsberichte des Wirtschafts- und Verkehrsministeriums, Nordrhein-Westfalen, Nr. 255. Westdeutscher Verlag, Köln und Opladen, 1956. 44 pp. DM 11.70.

The author shows that there is a group of oils whose viscosity decreases with temperature and can be represented by a formula, namely the reciprocal of the viscosity is proportional to the fourth power of the temperature difference, with two characteristic constants for each oil: a characteristic temperature where the viscosity is infinitely large and a proportionality constant. For such a hot liquid flowing through a duct, such as a circular pipe, it is shown that the oil cannot be cooled below a definite temperature, independent of the wall temperature. *Y. H. Kuo* (Peking).

Uchida, Shigeo. The pulsating viscous flow superposed on the steady laminar motion of incompressible fluid in a circular pipe. Z. Angew. Math. Phys. **7** (1956), 403–422.

An exact solution is presented for the pulsating flow of a viscous incompressible fluid in a circular pipe under the action of a pressure gradient with both steady and oscillating components. Since the flow is parallel to the axis, the basic equations are linear. The time-average of the total mass flow for the unsteady motion is found to be the same as for a steady flow with a pressure gradient equal to the mean pressure gradient. However, additional work is required to maintain this flow because of the additional dissipation associated with the oscillatory motion. The phase lag of the velocity oscillation from the pressure-gradient oscillation is found to increase from zero in the steady case to 90° when the frequency becomes infinite. Corresponding results are obtained for the instantaneous mass flow and surface friction. The conclusions are illustrated in all cases by detailed numerical calculations. *D. W. Dunn* (Ottawa, Ont.).

Nickel, K. Eine einfache dreidimensionale laminare Grenzschicht. Z. Angew. Math. Mech. **36** (1956), 303–304.

The author considers the class of exact solutions of the Navier-Stokes equations for which the boundary conditions and externally imposed body forces are independent of z, and u, v, and p at the boundary are not coupled with w. The solution under these conditions is independent of z, and u, v, and p satisfy equations that are independent of w. Thus w can be obtained from a linear differential equation after the previous quantities are determined. This independence principle had been considered previously only for the boundary-layer equations. The detailed nature of the solution for w is discussed in three simple cases for which the flow is stationary and $v=\text{const}=v_0$, namely, (a) $u(y)=\alpha+\beta y$, $v_0=0$; (b) $u(y)=\gamma+\delta y^2$, $v_0=0$; (c) $u=U_0\{1-\exp(v_0 y/\nu)\}$. *D. W. Dunn* (Ottawa, Ont.).

Shercliff, J. A. Entry of conducting and non-conducting fluids in pipes. Proc. Cambridge Philos. Soc. **52** (1956), 573–583.

This paper investigates the laminar flow of an electrically conducting, viscous fluid in the entry region of a pipe under a transverse magnetic field. The problem is linearized by a Rayleigh approximation, and the difference between the initial and ultimate velocity profiles is expressed as a series of orthogonal, exponentially decaying terms. The most persistent terms are calculated for the flow between two infinite parallel planes and in rectangular tubes when magnetic effects are present. Calculations are also made for the flow in circular pipes when magnetic effects are absent. The results for the last case are compared with those derived by other methods, in order to estimate the accuracy of the present method, and are found to be of the right magnitude. *D. W. Dunn* (Ottawa, Ont.).

Takaisi, Yorisaburo. The drag on a circular cylinder placed in a stream of viscous liquid midway between two parallel planes. J. Phys. Soc. Japan **11** (1956), 1092–1095.

Two values for the drag, on a circular cylinder placed, in a stream of viscous fluid, between two parallel walls, have been given previously. Bairstow, Cave, and Lang [Proc. Roy. Soc. London. Ser. A **100** (1922), 394–413] gave $2.26\pi\mu$ and Harrison [Trans. Cambridge Philos. Soc. **23** (1924), 71–88] gave $5.21\pi\mu$, where μ is the coefficient of viscosity. The author of the present paper uses an approximate method which is accurate to terms in $(a/b)^2$, where a and b are the radius of the cylinder and semi-width of the channel, respectively. A system of images is obtained which produces a flow satisfying the viscous condition of zero slip on the cylinder and walls. The value found by this method is $5.14\pi\mu$ and the author states that the difference between this value and that of Harrison is due to the fact that Harrison satisfies the viscous

boundary condition on the wall only along a few lines parallel to the axis of the cylinder. *G. N. Lance.*

★**Goldstein, S. Some developments of boundary layer theory in hydrodynamics.** Lecture Series, no. 33, prepared by W. H. Chu and S. I. Pai. Inst. for Fluid Dynamics and Appl. Math., Univ. of Maryland, 1955. i+27 pp.

In this lecture, Professor Goldstein reviews briefly some recent developments in the boundary-layer theory. The topics covered are as follows: solutions in Cartesian coordinates, the boundary-layer equations and solution in parabolic coordinates; higher approximations; the neighborhood of the leading edge and flows with wakes and separation. *Y. H. Kuo* (Peking).

★**Baron, Judson R. The binary-mixture boundary layer associated with mass transfer cooling at high speeds.** Technical Report 160. Massachusetts Institute of Technology, Naval Supersonic Laboratory, 1956. 280 pp.

Aerodynamic heating due to viscous dissipation in the boundary layer becomes of great importance at high speeds. The present report gives a comprehensive analysis of the effects of injecting a cooling fluid through the surface when the properties of this fluid are different from those of the fluid in the external stream. The report begins with a survey of the physical basis of the problem of the flow of gaseous mixtures, and then proceeds to a detailed derivation of the general flow equations for the case of two components. Simplifications are obtained by applying approximations of the standard boundary-layer type, but because of the physical complexity of the problem the final equations are still relatively complicated.

Some results of a general nature are obtained first. It is shown that the most significant parameter describing the mixture effects is the ratio of the molecular weights of the components (or equivalently, the ratio of the specific heats). The question of proper boundary conditions is carefully examined, and it is found that the mass injection rates and concentration conditions at the surface are related and so cannot be specified independently. The concept of the adiabatic-wall (or recovery) temperature is extended to the case of mixture flows by arbitrarily defining it as the surface temperature for which the only energy at the surface is that due to mass transfer. An order-of-magnitude analysis of the terms in the equations and boundary conditions depending on thermal diffusion and the related diffusion-thermo effect shows that they are relatively small in most cases of interest, although they could have a large effect on the recovery factor at low Mach numbers.

The remainder of the report deals with the solution of the equations in specific cases. The assumption of Prandtl and Schmidt numbers of unity, which is the correct order of magnitude for gases, leads to a simple solution that displays some of the qualitative features of the problem. For zero free stream pressure gradient, a generalization of the familiar quadratic relation between temperature and velocity indicates that with relatively less dense cooling fluids the heat transfer coefficient is reduced. However, the recovery factor remains unity as heretofore. For arbitrary Prandtl and Schmidt numbers, the equations are reduced to ordinary differential equations, and elaborate numerical calculations are performed for both helium and carbon dioxide injected into an air stream. With relatively less dense cooling fluids, there are again reductions in the skin friction and heat and mass transfer coefficients and also in the recovery factor, for a given mass-injection rate. Thus the use of a low density cooling fluid is most advantageous. An integral method of solution is devised for more general situations involving arbitrary variations of the free-stream pressure gradient and surface mass-transfer rate. Application of this to a problem in which injection takes place only over a finite length at the leading edge shows that significant mixture effects take place over several injection lengths downstream.

In most of the calculations, thermal diffusion effects are neglected, and it is assumed that the viscosity of the main-stream component is linearly related to the temperature. It is also assumed that the ratio of the mixture viscosity to the viscosity of the main-stream component, the corresponding ratio of specific heats, the Prandtl number, and the Schmidt number are dependent on the concentration only. The slight temperature dependence of the latter quantities and thermal diffusion effects are included in more accurate calculations for one case, but the changes are found to be small. *D. W. Dunn.*

Stefaniak, H. St. Die vollständigen Integrale der Grenzschicht-Differentialgleichungen für den runden und ebenen laminaren Strahl. Z. Angew. Math. Mech. **36** (1956), 310–311.

The boundary-layer equation for the round jet has the well-known form $FF'/\eta^2-F'^2/\eta-FF''/\eta=(F''-F'/\eta)'$, where $F(\eta)$ is a non-dimensional stream function and η is a non-dimensional distance normal to the axis. By the transformation $F=2\{(\eta/\varphi)(d\varphi/d\eta)+1\}$, it is shown that $\varphi(\eta)$ satisfies an equation whose general solution can be expressed in terms of the Bessel function $Z_p(a\eta)$, where a and p are arbitrary constants of integration. For the plane jet, the boundary-layer equation for the non-dimensional stream function $f(\eta)$ is $f'^2+ff''+f'''=0$. The transformation $f=(2/\psi)d\psi/d\eta$ then leads to an equation for ψ whose general solution can be expressed in terms of $\sqrt{(\eta-b)}Z_{\frac{1}{3}}(\frac{2}{3}a\sqrt{(\eta-b)^3})$. *D. W. Dunn* (Ottawa, Ont.).

Morgan, George W.; and Warner, W. H. On heat transfer in laminar boundary layers at high Prandtl number. J. Aero. Sci. **23** (1956), 937–948.

This paper is devoted to the asymptotic theory of the heat transfer through laminar boundary layers as the Prandtl number $\sigma\to\infty$. The theory for two-dimensional "forced convection" with arbitrary wall temperature was given by the reviewer [Proc. Roy. Soc. London. Ser. A. **202** (1950) 359–377; MR **12**, 218], who used the fact that in this limit the velocity distribution in the thermal boundary layer can be taken as a linear function of distance from the wall. The present authors discuss some particular examples, and then make further use of this result to treat free-convection flows and heat transfer through the boundary layer on a rotating disc.
M. J. Lighthill (Manchester).

Yasuhara, Michiru. On the hypersonic viscous flow past slender bodies of revolution. J. Phys. Soc. Japan **11** (1956), 878–886.

The author treats the hypersonic boundary layer in the case of a slender body of revolution by a method similar to that used by Stewartson [J. Aero. Sci. **22** (1955), 303–309; MR **16**, 877] for slender two-dimensional shapes. The problem is harder owing to the axisymmetry,

especially because the body radius is taken comparable with the thickness of the boundary layer, as well as with the thickness of the inviscid layer between boundary layer and shock wave. In the inviscid layer the entropy distribution has to be taken to be functionally related to the Stokes stream function in a manner dependent on the shock wave shape, and the pressure can vary normal to the stream. The latter variation is absent in the boundary layer, where however viscosity is not negligible.

The author seeks a particular solution for which the ratios of the thickness of the two layers to the body radius each remains constant with distance along the body. This is found to occur for body shapes with $r \propto x^{3/4}$. For these shapes solutions are obtained in both layers by solving ordinary differential equations. Valuable numerical results are derived, and discussed in details as well as illustrated graphically. *M. J. Lighthill.*

Rott, Nicholas; and Smith, William E. Some examples of laminar boundary-layer flow on rotating blades. J. Aero. Sci. **23** (1956), 991–996, 1006.

The equations for the approximate calculation of laminar boundary-layer flow on steadily rotating cylindrical blades were given by Fogarty [same J. **18** (1951), 247–252; MR **14**, 328.] These equations apply to regions at large radii from the axis of rotation, which is normal to the cylindrical axis. Here the cases are worked out for which the "chordwise flow" (cross-flow) is of the Falkner-Skan type. The "spanwise flow" (axial component) is calculated by solving a linear differential equation. It is pointed out that this equation can be separated into two parts, corresponding to its two inhomogeneous terms or, what is equivalent, the two terms in the formula for the axial component in potential flow about a rotating cylindrical blade [Sears, ibid. **17** (1950), 183–184; MR **11**, 623]. These are solved separately. It is proposed to use these results to construct an approximate procedure for arbitrary shapes. *W. R. Sears.*

Krzywoblocki, M. Z. Sur la turbulence spatialement homogène d'un fluide compressible. Publ. Sci. Tech. Ministère de l'Air, Paris no. 314 (1956), iv+54 pp. 650 francs.

L'auteur étend aux fluides turbulents compressibles les résultats généraux de la théorie statistique de la turbulence. Dans le chapitre I, il résume la théorie mathématique des fonctions aléatoires, et les principes de son application à l'hydrodynamique, tels qu'ils sont exposés par J. Kampé de Fériet [A. Blanc-Lapierre et R. Fortet, Théorie des fonctions aléatoires, Masson, Paris, 1953; MR **15**, 883]. Le chapitre II est consacré à la cinématique de la turbulence spatialement homogène d'un fluide compressible. Le fluide est défini par un champ scalaire-vectorial aléatoire, comportant la vitesse u_i, la masse volumique $\tilde{\omega}$ et la température θ. La pression et les diverses sortes d'énergie attachées aux fluides en résultent. De nombreuses valeurs moyennes sont introduites, entre autres:

$$\rho_{j,k}^{(u)}(h)=\overline{u_j(x',\omega)u_k^*(x,\omega)},$$

$$\rho^{\Gamma}(h)=\overline{\tilde{\omega}^{\frac{1}{2}}(x',\omega)\theta^{\frac{1}{2}}(x',\omega)\tilde{\omega}^{*\frac{1}{2}}(x,\omega)\theta^{*\frac{1}{2}}(x,\omega)},$$

$$\rho_{j,k}^{(u)}(h)=\overline{\tilde{\omega}^{\frac{1}{2}}(x',\omega)u_j(x',\omega)\tilde{\omega}^{*\frac{1}{2}}(x,\omega)u_k(x,\omega)}.$$

Ce sont des fonctions du vecteur $h=x'-x$. Dans ces formules, ω est un point de l'espace abstrait sur lequel est définie la mesure de probabilité. Ces moyennes se prêtent à l'analyse harmonique. On désigne en particulier par $\varphi_{j,k}^{(u)}(\lambda)$, $\varphi^{\Gamma}(\lambda)$ les fonctions dont $\rho_{j,k}^{(u)}(h)$, $\rho^{\Gamma}(h)$ sont les transformées de Fourier (dans le cas absolument continu). A ces fonctions spectrales, on peut rattacher l'énergie cinétique moyenne $\bar{E}(\omega)$ et l'énergie interne moyenne $\bar{E}_i(\omega)$ du fluide contenu dans un boîte B de volume 1.

Au chapitre III, les concepts dynamiques s'introduisent par l'intermédiaire des équations de Navier-Stokes (y compris l'équation thermodynamique). Le traitement statistique de ces équations montre en particulier que la masse moyenne et l'énergie moyenne totale (cinétique+interne) contenues dans la boîte B sont indépendantes du temps. Si l'on suppose l'indépendance entre les fonctions aléatoires $\tilde{\omega}$, u_j et β (coefficient de viscosité), on peut aller plus loin et démontrer en particulier que, si $\tilde{\theta}(\lambda, t)$ est la „fonction balance" qui donne la balance entre l'énergie cinétique par unité de masse et l'énergie dissipée par la viscosité et le travail de dilatation dans un élément de l'espace des fréquences Λ, alors $\int_{\Delta}\tilde{\theta}(\lambda,t)d\lambda=0$. Si l'on suppose que $\tilde{\theta}(\lambda,t)=0$, on peut en déduire quelques conséquences sur l'évolution des fonctions spectrales en fonction du temps. Il est enfin possible d'écrire la forme générale du tenseur spectral associé au tenseur de corrélation

$$\rho_{j,k}^{(p)}(h,t)=\overline{\tilde{\omega}(x',t,\omega)\tilde{\omega}^*(x,t,\omega)u_j(x',t,\omega)u_k^*(x,t,\omega)}.$$

J. Bass (Paris).

Ashley, Holt; and Zartarian, Garabed. Piston theory — a new aerodynamic tool for the aeroelastician. J. Aero. Sci. **23** (1956), 1109–1118.

Paper presents an interesting discussion of the application to aeroelastic problems of an aerodynamic theory valid for high Mach numbers. The method is based on a physical model originally proposed by Hayes and Lighthill which employs expansions in terms of the thickness and the inverse of the Mach number. The theory enables one to predict a point-function relationship between the local surface pressure and the normal velocity component. Method is able to account for certain nonlinearities in aerodynamic analyses. *H. N. Abramson.*

Storchi, Edoardo. Comportamento dei fluidi perfetti in regioni a velocità sonica. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19**(88) (1955), 389–404.

In this paper all possible steady two-dimensional nonviscous barotropic gas motions, rotational and irrotational, in which the local Mach number is everywhere 1, are found. *M. J. Lighthill* (Manchester).

Pai, S. I. Laminar jet mixing of two compressible fluids with heat release. J. Aero. Sci. **23** (1956), 1012–1018.

The problem of ignition and combustion in a mixing zone was previously considered by F. E. Marble and T. C. Adamson, Jr. [Jet Propulsion **24** (1954), 85–94]. In this paper, the author formulates the jet mixing problem with heat release in a general manner which includes the problem of Marble and Adamson as a special case. It is a curious fact that in this problem ignition of a combustible gas always starts for all temperature differences. This difficulty remains in the new paper. *Y. H. Kuo.*

Fowell, L. R. Exact and approximate solutions for the supersonic delta wing. J. Aero. Sci. **23** (1956), 709–720, 770.

It has become clear that an increasing use will have to be made of finite-difference schemes for solving problems

of steady three-dimensional supersonic inviscid flow over wing shapes. The paper under review might well be taken as a model of how to conduct such an investigation, particularly in the way that use is made of the computations of flow fields.

The problem treated is that of the delta wing of zero thickness with supersonic leading edges at angle of attack, for which the cone-field flows over expansion surface and compression surface may be treated separately as quite unconnected with each other.

On the expansion surface the main computation has to be done in a region of "cottage-loaf" shape bounded by different uniform flows separated from each other by a Prandtl-Meyer expansion fan. The boundary, between these uniform flows and the fan on the one hand, and the region of more complicated flow on the other, can be determined analytically. The authors' figure 4 shows how enormously it differs from that postulated by linear theory at an angle of attack 4° for Mach number 3 and sweepback angle 45°. The relaxation method is used to find a solution within this boundary. An interesting feature of the solution for f (the disturbance potential in cone-field coordinates) on the expansion surface itself is that it is very close to the "shifted linear solution" obtained by reducing the linear-theory value of f by a constant, chosen to make f satisfy the exact boundary condition.

At higher angles of attack the calculated boundary crosses itself and so no continuous solution is possible. It is assumed that this is the angle of attack at which an "inboard shock wave" would form on the surface, and the position of this shock wave is guessed on the assumption that it turns the surface component of the flow behind the Prandtl-Meyer fan into the main-stream direction. An experimental study is described which verifies the existence of such a shock wave at such a position.

On the compression surface the relaxation process has to be applied to the position of the boundary shock wave as well as to the equations of a motion which is no longer irrotational. Under these circumstances a good first guess at the flow field is invaluable, and here the authors are well served by a shifted linear solution of the type described above. The solution discovered is, in fact, once more close to that obtained by adding a constant to the linearised f, provided that the latter is extrapolated to make it possible to apply the condition at the boundary, which on the compression surface is of course outside the linear-theory position.

In each case the shift in f implies a uniform shift in the velocity component parallel to the free stream, without changes in the other components. If other exact solutions confirm the value of this shift, it may prove very useful to designers.

The one quantity that is adequately represented by the unmodified linear theory is the total integrated lift, due to cancellation of errors on the compression and expansion surfaces; in the example with $M=3$ noted above the linear-theory lift was in error by only 2 per cent, although pressures were out by 34 per cent.

M. J. Lighthill (Manchester).

Graham, E. W.; Lagerstrom, P. A.; Licher, R. M.; and Beane, B. J. A theoretical investigation of the drag of generalized aircraft configurations in supersonic flow. NACA Tech. Memo. no. 1421 (1957), iv+108 pp.

After a discussion of several useful elementary solutions of the equation of linearized supersonic small-perturbation flow of an inviscid fluid, the authors proceed to the evaluation of drag of distributions of singularities. From a general theorem the familiar formula of von Kármán for the drag of a slender body of revolution is derived and is extended to the general three-dimensional case, which yields Hayes' result [North American Aviation, Inc., Los Angeles, Reprt. no. A.L. 222, June 1947]. At this point, extensions are made for lift elements, leading-edge suction, loading discontinuities, and interference of elements. R. T. Jones' criteria for minimum drag [J. Aero. Sci. **18** (1951), 75–81; **19** (1952), 813–822; MR **12**, 554; **14**, 511] are given. Some examples of optimum distributions of lifting elements and of volume elements are shown. After a discussion of some questions regarding the uniqueness of minimum-drag solutions, the question of separability of lift, thickness and sideforce problems is considered. *W. R. Sears* (Ithaca, N.Y.).

Germain, P.; et Vallée, D. Effet de dièdre sur une aile delta en régime supersonique. Rech. Aéro. no. 52 (1956), 13–20.

Cone field theory is applied to the calculation of the forces acting on a Delta wing with dihedral in (linearized) supersonic flow (quasi-subsonic case). The incidence is supposed small but the angle of dihedral may be large. The method is extended to the case of three or more Delta wings joined along a common axis, as may be the case for a tail-unit. As pointed out by the authors, the case of ordinary dihedral has been treated also by J. B. L. Powell by a different method [Quart. J. Mech. Appl. Math. **9** (1956), 51–74; MR **17**, 914]. *A. Robinson.*

Whitham, G. B. On the propagation of weak shock waves. J. Fluid Mech. **1** (1956), 290–318.

The author's previous work on "improving" the linearized theory of compressible flow so as to make it capable of predicting the positions and strengths of any shock waves that appear [Comm. Pure Appl. Math. **5** (1952), 301–348; MR **14**, 330] is extended to problems which owing to the absence of any simplifying symmetry are problems in three, or even four, independent variables. To obtain the asymptotic form of the shock waves, and the behaviour at all times of the front shock wave, he shows that one may begin by adopting the asymptotic "geometrical-acoustics" approximation to the linearized solution. On this approximation wave energy is carried without attenuation along tubes made up of straight "rays". For each one-dimensional problem of wave propagation along one of these tubes (in which of course variation of the cross-sectional area A of the tube with distance s along it must be taken into account) the "improvement" of the linear theory to take into account nonlinearity to a first approximation (by correcting the velocity of the wave) proceeds as in the author's earlier theories. The shock wave position and strength follow from the shock conditions, also as before. As a sample of the results, the strength of the front shock wave is found to vary with s like

$$(A\int_0^s A^{-1}ds)^{-1}.$$

The theory as it stands is inapplicable to problems where A becomes zero anywhere (this is the condition satisfied at a caustic). The author refers to a forthcoming paper as dealing with such cases.

Particular cases treated include unsymmetrical ex-

plosions resulting from the sudden release of excess pressure in a region of non-spherical shape, the theory of "supersonic bangs" produced in the unsteady motion of aeroplanes, and the steady shock-wave patterns of various kinds of thin three-dimensional wing shapes. The last application gives a novel approach to the so-called "supersonic area rule". *M. J. Lighthill.*

Miles, John W. The aerodynamic force on an airfoil in a moving gust. J. Aero. Sci. 23 (1956), 1044–1050.

This is an application of classical two-dimensional thin-airfoil theory to the problem of an airfoil flying at constant speed U, over which a sharp-edged gust (upwash field) passes with relative speed $\lambda^{-1}U$, λ=const. Wagner's problem of sudden change of incidence is given by $\lambda=0$, and the classical sharp-edged gust problem is given by $\lambda=1$. Laplace transforms are employed; the inverse transforms involved are calculated with the same sort of approximation as Jones' [NACA Rep. no. **681** (1940)]. Eight values of λ are given, in the range $-1\leq\lambda\leq+1$.

The "virtual mass component" must be added to these results; this gives the curves of total lift against time (distance) a curious appearance for negative λ (gust overtaking airfoil from rear.) {The reviewer recalls that Küssner originally erred in calculating the classical case $\lambda=1$ [Luftfahrtforschung **13** (1936), 410–424] and actually obtained the result given here for $\lambda=-1$. The hump-backed shape of the curve led him to the proposal that the "virtual mass" component be discarded, a suggestion which he subsequently retracted.} *W. R. Sears.*

Evans, Cerda; and Evans, Foster. Shock compression of a perfect gas. J. Fluid Mech. **1** (1956), 399–408.

Un fluide compressible est initialement au repos entre une paroi fixe et un piston mobile. Le piston est animé d'une vitesse constante; un choc prend naissance et se réfléchit successivement sur la paroi et sur le piston. L'auteur établit les valeurs de le pression et de le température après un nombre arbitraire n de réflexions; il calcule le valeur limite de l'entropie lorsque n augmente indéfiniment. *H. Cabannes* (Québec).

Freeman, N. C. On the theory of hypersonic flow past plane and axially symmetric bluff bodies. J. Fluid Mech. **1** (1956), 366–387.

L'auteur étudie les écoulements hypersoniques par une méthode d'approximations successives susceptible d'être utilisée dans de nombreuses questions. La première approximation correspond aux fluides pour lesquelles le rapport des chaleurs spécifiques posséderait le valeur constante $\gamma=1$. Pour les petites valeurs du paramètre $\varepsilon=(\gamma-1)/(\gamma+1)$ des approximations successives sont possibles et les deux premières (termes indépendants de ε et termes en ε) sont explicitées. Les lignes de courant, l'onde de choc, et la distribution des pressions sont déterminées dans plusieurs cas. *H. Cabannes* (Québec).

Riabouchinsky, Dimitri. Sur le paradoxe d'Ackeret de la portance indéfiniment grande d'une plaque au voisinage de la valeur unité du nombre de Mach. C. R. Acad. Sci. Paris **244** (1957), 281–284.

Rott, Nicholas. Diffraction of a weak shock with vortex generation. J. Fluid Mech. **1** (1956), 111–128 (1 plate).

This is a theoretical discussion of the vorticity produced at the corner of a wedge when a weak shock wave is diffracted round it. The incompressible-flow equations may be used in the region near the corner where the vorticity is, and where the Sommerfeld solution for diffraction with irrotational flow may be approximated by a simple power-law solution satisfying the incompressible-flow equation and giving infinite velocity at the corner. To this solution in this region must be added a rotational flow field cancelling out the infinity and tending to zero far from the corner. From dimensional considerations the dependence of this rotational field on the velocity u_0 behind the incident shock wave is found; distance of any vortex element from the corner after time t is proportional to

$$(1) \qquad at\left(\frac{u_0}{a}\right)^{(3\pi-\beta)/(3\pi-2\beta)},$$

where a is the velocity of sound and β is the angle of the wedge. This prediction has been verified by Howard and Matthews [J. Appl. Phys. **27** (1956), 223–231].

To proceed further the author uses a model involving a single line-vortex, applying a condition due to Brown and Michael [J. Aero. Sci. **21** (1954), 690–694, 706] on the force sustained by this vortex, to determine its distance from the corner, or rather the factor of proportionality that must be applied in (1) above, to within 10% of experimental values. Its angular position is predicted less well, and it is shown that a vortex-spiral model would be needed to improve the position here; some conditions, that such a model would have to satisfy, are found. *M. J. Lighthill* (Manchester).

See also: Misicu, p. 470; Borg, p. 474; Heins, p. 478; Morton, p. 518; Neumark, p. 523; Pearcey and Hill, p. 524; Millsaps and Pohlhausen, p. 538; McVittie and Rogers, p. 544; Ruzin, p. 545.

Optics, Electromagnetic Theory, Circuits

★ **Flügge, Johannes. Leitfaden der geometrischen Optik und des Optikrechnens; mit praktischen Aufgaben.** Vandenhoeck & Ruprecht, Göttingen, 1956. 202 pp. DM 19.80.

The aim of the author is to give a treatise of elementary geometrical optics, which is understandable to a technically trained user of optical instruments.

From the vector form of Snell's law, the author develops the formula of paraxial optics in great detail. with numerical examples and calculation schemes for first-order approximation optics.

The third chapter contains an especially lucid discussion of the problems of vignetting, i.e., of the consideration of determining the limiting rays which actually traverse the system, as far as such a discussion is valid, using only Gaussian approximation.

The fourth chapter discusses in the usual manner the third-order aberrations and generalizes them to systems with finite aperture and field. A warning signal that these are not the only errors to be considered in such a system is missing.

The fifth chapter deals with some of the color aberrations. The very short sixth chapter gives without derivations formulae for third-order aberrations.

The seventh chapter gives as an example a numerical ray trace of meridional rays through a triplet for three colors and the corresponding error curves.

The eighth chapter gives a number of practical problems which are all of general optical interest.

The book is an elementary treatise and highly recommendable for engineers and physicists who want to build instruments in which optical devices are used.
M. Herzberger (Rochester, N.Y.).

Grümm, H. Ein streng auswertbares Modell für elektronische Immersions-Zylinderlinsen. Optik **13** (1956), 277–288.

In this paper the author investigates the focusing properties of an electron beam in a special two-dimensional electrostatic field for which the Hamilton-Jacobi equation becomes separable in elliptic coordinates. The equations of the paths are expressed in an exact form in terms of Legendre elliptic functions involving a parameter λ which depends on the so-called "power" of the lens. The domains of actual paths of a parallel electron beam are analyzed and various lens properties of such a field are derived, in particular the conditions for the mirror properties of the lens. Then follows a treatment of the paraxial equation which yields to exact solutions in terms of elliptic functions. The focal and other properties of such an electron lens model are discussed in detail. A number of curves representing the paths of the electrons with various initial data are included in the text. [See also the following review.] *N. Chako.*

Grümm, H.; und Kremser, H. Beispiele streng berechenbarer Ionen- und Elektronenbahnen im ebenen elektrostatischen Feld. Optik **13** (1956), 295–306.

In this paper the authors have calculated the trajectories of charged particles for a two-dimensional electrostatic field produced by cylindrical type condensors. By employing circular and elliptical (cylinder) coordinates, the Hamilton-Jacobi equation becomes separable and the solutions describing the paths have been expressed in exact form by elliptic functions. Various limiting cases are discussed for which the solutions reduce to elementary functions. The paraxial equation of the charged particles has also been solved for the same potential fields. The authors discuss in some detail the allowed regions where the charged particles actually move, including the special case of a parallel beam impinging normally to the field (x-y plane). The results of calculations of the potential fields are presented in several graphs, including the paths of the charged particles for various initial data. *N. Chako.*

Glaser, Walter. Elektronische Abbildung als Eigenwertproblem. Österreich. Ing.-Arch. **10** (1956), 167–171.

The problem of focusing paraxial electrons by an arbitrary rotational magnetic field has been formulated as an eigenvalue problem in terms of a parameter k, which represents the power (strength) of the lens. From the paraxial equation (i) $r''+k^2F(x)r=0$, subject to the boundary conditions (b.c.), $r(x_0)=r(x_1)=0$, where x_0, x_1 are the positional coordinates of the object and image, the author determines k by means of Rayleigh's variational method, i.e. by minimizing k^2 with respect to another parameter of the lens, namely, its magnification. If the field is of a bell shape type, then (i) has exact solutions and k is expressed in a simple form in terms of the magnification of the lens. For arbitrary fields, the first approximation to the exact value of k is obtained by using the solution of the former problem in the Rayleigh formula. A more accurate value of k is obtained by using a two-term Fourier series for the solution of (i) involving an arbitrary constant in the Rayleigh formula (Ritz procedure) and, then minimizing k^2 with respect to the magnification parameter of the lens, β. This procedure leads to a quadratic equation in β, from the solutions of which, $k_{\min}$ is evaluated in terms of known quantities obtained from the bell-shaped field problem. *N. Chako.*

Vandakurov, Yu. V. Electron-optical systems whose fields do not depend on one coordinate. Ž. Tehn. Fiz. **26** (1956), 2578–2594. (Russian)

Želudev, I. S. Dielectric ellipsoids and surfaces of dielectric permeability. Akad. Nauk SSSR. Kristallografiya **1** (1956), 105–111. (Russian)

Let $\mathbf{P}=\Pi_e\mathbf{E}$, $\mathbf{D}=\Pi_e\mathbf{E}$ be the constitutive relations for an electric field in an anisotropic medium. The author gives a detailed exposition of the geometrical relation between the ellipsoids $\mathbf{E}\cdot(\Pi_e\mathbf{E})=1$, $\mathbf{D}\cdot(\Pi_e^{-1}\mathbf{D})=1$ and the field vectors **E**, **D**, and **P**. Also constructed are two "surfaces of dielectric permeability", obtained by marking off along **D**, **E**, drawn in all possible directions from a point, the lengths $\varepsilon_N=D^2/(\mathbf{D}\cdot\mathbf{E})$, $\varepsilon_S=(\mathbf{D}\cdot\mathbf{E})/E^2$, respectively. *F. V. Atkinson* (Canberra).

Jansen, L.; and Solem, A. D. Molecular theory of the dielectric constant of nonpolar gases. Phys. Rev. (2) **104** (1956), 1291–1297.

In order to explain the measured deviations from the Clausius-Mosotti relationship between the dielectric constant of a gas and the polarizability of a free molecule, a theory of molecular polarizability is developed taking into account the interaction between molecules. It is assumed that this interaction contains induced dipole and induced and permanent quadrupole moments. The calculation is restricted to a system of axially symmetric molecules in which the density is low enough so that no more than two molecules interact simultaneously. The multipole expansion is given in tensor notation and then the induced dipole moment is calculated. The first order perturbation calculation gives the usual result of the free molecule's polarizability times the field strength. Then the second and third orders are calculated, the latter comprising 29 distinct terms. The result is also expressed in terms of an average "effective" polarizability. The analogous expressions involving higher multipoles and the revised form of the Clausius-Mosotti function are promised in a subsequent paper. *M. J. Moravcsik.*

Štepa, N. L. Grapho-analytical construction of spatial trajectories of charged particles in electrostatic fields by the method of radii of curvature. Ž. Tehn. Fiz. **26** (1956), 2281–2286. (Russian)

The author extends the grapho-analytical method of radii of curvature for finding the trajectories of charged particles in a plane to the construction of spatial trajectories by means of projections on two coordinate planes. He further generalizes it to take account of relativistic effects. A numerical example is presented wherein the values attained by this method agree qualitatively with the true values, while the quantitative error is reasonably small. *P. Rabinowitz* (Rehovoth).

Infeld, L.; and Plebański, J. A simple derivation of the equations of motion in classical electrodynamics. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 347–351.

The field quantities are expressed in powers of c^{-1} and by making effective use of the spherical symmetry of the

three-dimensional delta function, the equations of motion of a classical point electron, including the radiation term, are obtained. *A. J. Coleman* (Toronto, Ont.).

Hofmann, H. Über den Kraftangriff des stationären elektromagnetischen Feldes an der Materie. Österreich. Ing.-Arch. **10** (1956), 393–415.

Wait, James R. Shielding of a transient electromagnetic dipole field by a conductive sheet. Canad. J. Phys. **34** (1956), 890–893.

By applying Laplace transformation to earlier results of his own in the steady-state case [Appl. Sci. Res. B. **3** (1953), 230–236], the author obtains simple closed-form expressions for the field of a transient dipole in the presence of a conducting sheet. *C. J. Bouwkamp.*

Chambers, Ll. G. Reflection of a wave by a cylindrical mirror. Proc. Edinburgh Math. Soc. (2) **9** (1956), 145–150.

The author presents a straightforward solution to the problem for an arbitrarily polarized plane wave at an arbitrary angle of incidence upon the infinite cylinder. He shows that in general *TE* and *TM* mode are excited. One graph of numerical results for mode intercoupling is included. *W. K. Saunders* (Washington, D.C.).

Wu, Tai Tsun. High-frequency scattering. Phys. Rev. (2) **104** (1956), 1201–1212.

A method for treating the two-dimensional diffraction of a plane wave by a cylinder is reported. The doubly connected physical space is replaced by the simply connected many-sheeted universal covering space. In this space the wave equation is solved with appropriate boundary conditions. The solution in each sheet is called a 'creeping wave', and the actual solution is obtained by adding them. As $\lambda \to 0$ the successive terms of this series decrease exponentially, so that for the asymptotic expression the first term alone need be considered. The forward amplitude requires special treatment; explicit asymptotic expansions for it are obtained in the form of power series in $\lambda^{2/3}$. Finally the scattering by a sphere is treated in a formally analogous way, although the concept of creeping waves does not apply. The values for the scattering cross-sections obtained from the asymptotic expansions are compared with the exact solutions.

N. G. van Kampen (Utrecht).

Wait, James R. Radiation from a vertical antenna over a curved stratified ground. J. Res. Nat. Bur. Standards **56** (1956), 237–244.

The approach taken by the author is to substitute one impedance boundary condition for the multiple matching equations required otherwise at the various interfaces. This technique yields a dividend in the form of a more tractable contour integral, taken as usual over the index of the Legendre function. A method suggested to the author by H. Bremmer is used for the computation of the limiting case where the receiver is near the optical horizon. Some numerical results are presented.

The author wishes to note the following misprints: Eq. (10) – Part of Legendre index misplaced. Eq. (35) – Missing bracket after $M(g, \rho)$. Displayed line below Eq. (40) — Term exp $(-\rho)$ which should also be a multiplier of erfc has been omitted.

W. K. Saunders (Washington, D.C.).

De Socio, Marialuisa. Sulla velocità dell'energia in una guida d'onda soggetta ad un campo magnetico. Boll. Un. Mat. Ital. (3) **11** (1956), 566–568.

★ **Strieby, M. A Fourier method for time domain synthesis.** Proceedings of the Symposium on Modern Network Synthesis, New York, 1955, pp. 197–209. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1956.

"The aspect of time domain synthesis which will be considered here is that of approximating an impulse response."

"In this method one period of a Fourier series (or other trigonometric series) is used to approximate the desired time function. The series shows how many poles are needed in the network system function and gives one an idea of the error function to be expected. It can be adjusted to achieve a reasonable compromise between error and network complexity. Once this is done, a realizable system function is straightforwardly computed." *N. Levinson* (Cambridge, Mass.).

★ **Dantzig, G. B.; and Fulkerson, D. R. On the max-flow min-cut theorem of networks.** Linear inequalities and related systems, pp. 215–221. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

Let a current enter a graph at a vertex A and leave at a vertex B. Let each edge and vertex have a non-negative "capacity" which is not to be exceeded by the magnitude of the current through that edge or vertex. The authors solve the problem of finding the maximum possible flow through the graph from A to B, treating it as a problem in linear programming. Their theory includes the case of oriented graphs, where it is assumed that the current in any edge is from the negative to the positive end. Menger's theorem on graphs can be proved as a special case.

W. T. Tutte (Toronto, Ont.).

Pussèt, L. A. On a type of auto-oscillations in systems with regulated angular coordinate. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1956, no. **9**, 24–30. (Russian)

A study of the stability of a system characterized by the equation:

$$I\ddot{\varphi}+R\dot{\varphi}+K\varphi=F\sin(\omega t+\varphi),$$

where I, R, K, F, and ω are constants. Linearizing this equation and expressing its solution as a power series in a parameter, the author obtains a form of Hill's equation whose stability is investigated by a method analogous to that used by Stoker [Nonlinear vibrations in mechanical and electrical systems, Interscience, New York, 1950; MR 11, 666]. Stability regions for this equation are derived and stated in terms of the parameters of the original equation. *L. A. Zadeh* (New York, N.Y.).

Savinov, G. V. Self-vibrating systems with parametric self-excitation. Vestnik Moskov. Univ. **11** (1956), no. 6, 27–32. (Russian)

The author considers a second order system in which reactive parameters (such as time-constants of *RC* networks) are functions of the output and its derivative. For certain special forms of these functions, conditions are found under which the system is oscillatory.

L. A. Zadeh (New York, N.Y.).

Sideriadès, L. Systèmes couplés non linéaires. J. Phys. Radium (8) **17** (1956), 159A–175A.

L'auteur étudie les systèmes couplés non linéaires de

nature électronique et en fait une synthèse générale. Il retrouve comme cas particuliers les exemples classiques de la triode oscillatrice, du multivibrateur, de l'Ecclès-Jordan, etc. Les méthodes mathématiques utilisées relèvent de l'analyse topologique. Le domain d'application est limite au premier ordre pour les systèmes différentiels. *Résumé de l'auteur.*

Kline, Morris. Basic research in electromagnetic theory. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Final Rep. No. **EM-97** (1956), ii+61 pp.

This final report of a long series organizes the material presented in the earlier reports to make clear the overall objectives of the program. A brief discussion is given of the various mathematical techniques studied or developed for the treatment of the problems. Examples of such techniques are: asymptotic solutions of electromagnetic problems, the Wiener-Hopf methods, and the methods used to solve differential equations. A summary is also presented of the problems investigated by means of these techniques and of the results obtained. The fields of interest cover diffraction theory, antenna theory, and wave propagation. The specific problems deal with scattering from smooth bodies, diffraction by apertures, discs, and combination of obstacles, properties of antenna radiation patterns and the generation of these patterns by sources, tropospheric wave propagation in smooth media, ionospheric wave propagation, the inverse propagation problem, i.e. the determination of properties of the ionosphere from known results on wave propagation, scattering by turbulent air masses, and certain aspects of propagation in wave guides. A complete bibliography of work done on this project is included. *J. E. Rosenthal.*

Mattsson, Gösta. Distortionless coaxial cables. Ericsson Technics **12** (1956), 29–59.

The paper refers to the partial differential equations valid for homogeneous and loaded telephone lines.

The author determines the propagation constants for a homogeneous coaxial cable with a tubular centre conductor, taking skin effect into account. He proceeds to find the propagation on such a cable loaded at equidistant points with shunt inductances or series condensers by means of Laurent's frequency transformation method and shows that such a loaded cable can be so proportioned that distortionless transmission is obtained over a large frequency band extending from zero frequency upwards. In this case, the characteristic impedance can be represented over the entire frequency range by a resistor in series with a capacitor. *J. Rybner* (Copenhagen).

See also: Heins, p. 478; Pignedoli, p. 488; van den Dungen, Hontoy et Janssens, p. 497; Kessler, Prusch and Stegun, p. 517; Prokof'ev, p. 524; Glauberman, p. 524; Shercliff, p. 530; Baron, p. 531; Četaev, p. 545; Popov, p. 549; Moore and Shannon, p. 549.

Thermodynamics and Heat

Callen, Herbert B. Principle of minimum entropy production. Phys. Rev. (2) **105** (1957), 360–365.

The author attempts to prove the theorem of minimum entropy production for the steady state of an irreversible process using the quantum mechanical density matrix. However, he does not use the quantum mechanical equations of motion to solve for the steady state condition. Instead he treats the averages of the individual matrix elements of the density matrix as phenomenologically identifiable thermodynamic parameters. Then assuming an equilibrium, Gaussian, distribution for the elements of the density matrix, linear equations between fluxes and affinities, and the behavior of these matrix elements under momentum reversal, he is able to derive the Onsager reciprocity relations from which the minimum entropy principle follows in the same way as in the usual phenomenological theory. *D. Falkoff* (Waltham, Mass.).

Fieber, H. Über das Temperaturfeld in längs einer Richtung bewegten und zeitlich veränderlichen Bereichen. Österreich. Ing.-Arch. **10** (1956), 155–160.

When a metal rod is formed by extrusion from a die at a constant rate, the temperature of the metal at the die is higher than the temperature in the surrounding atmosphere, and so there is a temperature gradient in the rod. The problem is to find the temperature in the rod.

The paper is rather hard to follow, and the analysis has not been carried out to completion. The essential idea, in a slightly simplified notation, is as follows. If the temperature of the surrounding atmosphere is taken as zero, and if the axis of the rod is taken as axis of z, so that the end of the rod is at $z=vt$ at time t, where v is a constant, the temperature T in the rod satisfies

$$\frac{\partial T}{\partial t}=a^2\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)-v\frac{\partial T}{\partial z}\quad(0<z<vt)$$

(where a is a constant) under the following conditions: (i) $T=0$ $(t=0)$, (ii) $T=T_0>0$ $(z=0,\ t>0)$, (iii) $\alpha\,\partial T/\partial z+\beta T=0$ $(z=vt)$, (iv) a homogeneous boundary condition on the boundary of the uniform cross-section of the rod.

This is equivalent to solving

$$\frac{\partial \Theta}{\partial t}=a^2\nabla^2\Theta-v\frac{\partial \Theta}{\partial z}+q(x,y,z)\delta(z-vt)\quad(0<z<\infty)$$

where δ is Dirac's function and q is an unknown source function, under the conditions (i) (ii) (iv). If q is chosen so that Θ satisfies (iii), then $\Theta\equiv T$ in $0<z<vt$.

It is possible to expand Θ in terms of the eigenfunctions of the operator $\partial^2/\partial x^2+\partial^2/\partial y^2$ under the conditions (iv), i.e. to write $\Theta=\sum \vartheta_\kappa(z,t)\varphi_\kappa(x,y)$ where

$$\frac{\partial^2\varphi_\kappa}{\partial x^2}+\frac{\partial^2\varphi_\kappa}{\partial y^2}=\kappa\varphi_\kappa.$$

If we expand $q=\sum Q_\kappa(z)\varphi_\kappa(x,y)$, then

$$\frac{\partial\vartheta_\kappa}{\partial t}=a^2\frac{\partial^2\vartheta_\kappa}{\partial z^2}-v\frac{\partial\vartheta_\kappa}{\partial z}+a^2\kappa\vartheta_\kappa+Q_\kappa(z)\delta(z-vt).$$

This equation can be solved, under the appropriate boundary conditions, by standard techniques. The solution consists of known functions plus an integral involving Q_κ. It remains to find Q_κ so that (iii) is satisfied. This involves solving a Volterra integral equation of the first kind; but the solution is not carried to completion. *E. T. Copson* (St. Andrews).

Selig, F. Bemerkungen zum Stefanschen Problem. Österreich. Ing.-Arch. **10** (1956), 277–280.

Suppose that a semi-infinite prism of ice is contained in a cylinder with insulated walls. Initially the whole prism is at freezing point so that the temperature $T(x,t)$ satisfies the initial condition $T(x,0)=0$. If heat is applied at the end $x=0$ of the prism, the temperature satisfies an end condition $T(0,t)=\varphi(t)$ where φ is a given function.

The ice then gradually melts, and there is a moving boundary $x=x_1(t)$, separating water and ice, on which T vanishes. At this boundary, owing to the latent heat of fusion, $-\kappa\partial T/\partial x=\lambda dx/dt$. The problem is to find $T(x, t)$ to satisfy the equation of heat

$$\frac{\partial T}{\partial t}=a^2\frac{\partial^2 T}{\partial x^2}\quad(0<x<x_1(t))$$

under these conditions and also to find $x_1(t)$.

As in the paper reviewed above, the author considers instead the equation

$$\frac{\partial\vartheta}{\partial t}=a^2\frac{\partial^2\vartheta}{\partial x^2}-\lambda\frac{dx}{dt}\,\delta(x-x_1(t))\quad(0<\frac{x}{t}<\infty).$$

If we find the solution of this equation such that $\vartheta(x, 0)=0$ $\vartheta(0, t)=\varphi(t)$, $\vartheta(x_1(t), t)=0$, then, in $0<x<x_1(t)$, ϑ is identical with T.

In fact, we have

$$\vartheta(x, t)=-\frac{\lambda}{2a\sqrt{\pi}}\int_0^t\dot{x}_1(\tau)\left[\exp\left\{-\frac{[x-x_1(\tau)]^2}{4a^2(t-\tau)}\right\}-\exp\left\{-\frac{[x+x_1(\tau)]^2}{4a^2(t-\tau)}\right\}\right]\frac{d\tau}{\sqrt{(t-\tau)}}+\int_0^t\psi(\tau)\operatorname{erfc}\left(\frac{x}{2a\sqrt{(t-\tau)}}\right)d\tau$$

where $\psi(t)=d\phi/dt$. If we put $x=x_1(t)$, ϑ should vanish; this gives a complicated equation to determine x_1.

In the simplest case, $\psi(t)=\delta(t)$, $\varphi(0)=0$, it is known that $x_1=c\sqrt{t}$; and the method gives the equation for c found by Stefan [S.-B. Akad. Wiss. Wien. IIa. 98(1889), 473–484]. *E. T. Copson* (St. Andrews).

Hoff, N. J. Thermal buckling of supersonic wing panels. J. Aero. Sci. 23 (1956), 1019–1028, 1050.

In this paper the temperature and stress distributions are calculated in multicellular supersonic wing structures. A buckling criterion is established for the panels of cover plates subjected to thermal stresses.

Some important results of general interest may be quoted here, namely the temperature distribution in the cover plates and the buckling load.

If a series of identical cells is subjected to aerodynamic heating of a constant intensity q (per unit area per second), the temperature of the cover plates, in absence of shear webs, would rise in accordance with the equation (1) $T_c=Ht$, where T_c is the uniform temperature of the cover plates, t is time and (2) $H=q/h_c\rho_c c$ with h_c the thickness of the cover plate, c the specific heat and ρ_c the weight density of its material. Under most practical conditions, this is also in good agreement with the temperature in the cover plate midway between two adjacent shear webs. In the derivation of this formula it was assumed that the two cover plates were heated equally and that no heat was lost by the cover plates through radiation or convection.

At the juncture with the shear webs, the temperature of the cover plates is lower is consequence of the heat flowing into the web. The temperature is given by the equation

$$T_{cw}=(1-p^*)Ht,\tag{3}$$

where

$$1/p^*=1+(h_c/h_w)(K_c/L_w)(a_w/a_c)^{\frac{1}{2}},\tag{4}$$

K is the conductivity, a the diffusivity and h the plate thickness. The subscripts c and w refer to cover plate and web, respectively. When cover plate and web are of the same material, Eq. (4) reduces to (4a) $1/p^*=1+h_c/h_w$.

The second interesting result refers to plates in which the thermal stress distribution can be represented approximately by a uniform compressive stress upon which is superimposed a second harmonic of the compressive stress whose amplitude is p times the average stress. The buckling stress σ_{cr} of the long rectangular plate of width L_c and thickness h_c is given by

$$\sigma_{cr}=\frac{\pi^2E_c}{3(1-\nu_c^2)}\left(\frac{h_c}{L_c}\right)^2\Big/\left(1-\frac{p}{2}\right),\tag{5}$$

where E_c is the modulus of elasticity, ν_c Poisson's ratio and p the ratio of the second harmonic of the compressive stress to the average compressive stress. In this formula, σ_{cr} is the average compressive stress, i.e. the total compressive load acting on the plate divided by L_ch_c. In the development of this formula, the plate was assumed to be simply supported along the shear webs. Eq. (5) is sufficiently accurate for practical use when $-2<p<1$.

R. Gran Olsson (Trondheim).

★ **Millsaps, Knox; and Pohlhausen, Karl. Heat transfer to Hagen-Poiseuille flows.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 271–294. University of Maryland Book Store, College Park, Md., 1956.

The thermal distribution in the laminar flow of an incompressible viscous fluid in a circular tube is considered in the case when the fluid and wall of the tube are held at a constant temperature up to a certain cross-section at which the temperature of the wall is changed to a new value. In the steady case, certain terms in the heat-transfer equation corresponding to viscous dissipation and to heat conduction in the flow direction, which had been neglected in previous work on the problem, are now retained. The inclusion of the dissipation term is shown to result in a simple additive correction to the solution obtained otherwise. The remaining part of the solution is obtained as a series whose typical term is a product of an exponential function and a Bessel function of order zero, with the coefficients apparently determined from conditions obtained by equating to zero certain weighted averages of the basic equation. (Only a brief description of the method is given.) The retention of the mentioned heat-conduction term is found to be important for small values of the Péclet number. The unsteady case is considered with the neglect of the terms mentioned above. Detailed numerical results are presented in the form of tables and graphs. *D. W. Dunn.*

Cotte, M. Sur un problème de transport de chaleur par déplacement d'un solide. J. Phys. Radium (8) 17 (1956), 116A–120A.

Lifšic, I. M. On temperature outbursts in a medium subject to the action of nuclear emission. Dokl. Akad. Nauk SSSR (N.S.) 109 (1956), 1109–1111. (Russian)

In a body bounded by a surface Ω there appear instantaneous random point-sources of heat, with elementary probability n_0dVdt. The problem is to find under stationarity assumption the probability density $P(\theta)$ that at a given point $T-\overline{T}=\theta$, where T is the temperature and $\overline{T}=\overline{T}(\mathbf{r})$ is the mean temperature.

It is found that under some supplementary assumptions $P(\theta)$ is given by an expression of the form

$$(2\pi)^{-1}\int_{-i\infty+0}^{i\infty+0}\exp[z(T+\theta)-n_0\varphi(z)].$$

Various approximations are given when $n_0 \gg 1$; for example, the approximation $(5/3\pi\sqrt{3})\Gamma(2/3)\theta^{-3/2}$ when $n_0\theta \gg 1$. *M. Loève* (Berkeley, Calif.).

Jung, H. Zur Berechnung von Wärmeaustauschern. Österreich. Ing.-Arch. **10** (1956), 382–392.

Kogan, M. G. Non-stationary heat conduction in solids bounded by orthogonal coordinate surfaces. Ž. Tehn. Fiz. **26** (1956), 1185–1193. (Russian)

The author investigates the non-stationary three-dimensional temperature field in homogeneous and isotropic rigid bodies. The well-known heat-conduction equation is subjected to the elementary boundary-conditions as well as Sommerfeld's radiation condition. For mixed boundary-conditions, expansions in terms of eigen-functions are not particularly useful, and therefore the author introduces a method of Grinberg [Selected questions of the mathematical theory of electric and magnetic phenomena, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1948]. The calculated temperature variation at the surface and at the axis of a circular cylinder (steel) and the measured field, as plotted by the author, seem to be in good agreement. {The mathematical reasoning in the present work is not convincing; however, the author might give more explicit details in later communications.}

K. Bhagwandin (Oslo).

Mirzadžanzade, A. H., and Džalilov, K. N. On approximate solution of the one-dimensional problem of Stefan. Ž. Tehn. Fiz. **25** (1955), 1800–1803. (Russian)

This paper gives a method for finding approximate solutions of the one dimensional problem fo Stefan which requires functions T_1 and T_2 satisfying

$$(1)\qquad \frac{\partial^2 T_1}{\partial x^2} = a_1^2 \frac{\partial T_1}{\partial t} \quad (0<x<l,\ t>0),$$

$$(2)\qquad \frac{\partial^2 T^2}{\partial x^2} = a_2^2 \frac{\partial T_2}{\partial t} \quad (l<x<\infty,\ t>0),$$

and the conditions

$$(3)\quad T_1(0,t)=T_0=\text{const}, \qquad (4)\quad T_2(x,0)=0,$$

$$(5)\quad T_2(\infty,t)=0, \qquad (6)\quad T_1(l,t)=T_2(l,t)=T_{*\lambda}=\text{const},$$

$$(7)\qquad \lambda_2 \frac{\partial T_2}{\partial x}(l,t) - \lambda_1 \frac{\partial T_1}{\partial x}(l,t) = k\gamma_2 \frac{dl}{dt}.$$

The method is based on the assumption that condition (5) implies that an approximation for $T_2(x, L)$ must satisfy

$$(8)\quad T_2(L,t)=0, \qquad (9)\quad \frac{\partial T_2}{\partial x}(L,t)=0,$$

for $x=L$, large. The zeroth approximation is obtained by taking $\partial^2 T_1/\partial x^2=0$ and $\partial^2 T_2/\partial x^2=0$, integrating and making use of conditions (3), (6) and (8). This gives T_1 and T_2 as linear functions of x and unknown functions of L and l, which upon substitution into (1) and (2) gives expressions for the second derivatives of T_1 and T_2 in terms of x, L, l, L', l'. Conditions (7) and (9) then give a pair of ordinary differential equations for L and l as functions of time, or alternately a single differential equation for L as a function of l. This in turn leads to an approximate expression for l as a function of t through equation (8).

A statement is made to the effect that in practice the first order approximation is all that is needed but no discussion is given of the convergence question.

C. G. Maple (Ames, Ia.).

See also: Betts, Bhatia and Wyman, p. 475; Lessen, p. 524; Trlifaj, p. 525; v. Schlippe, p. 530; Morgan and Warner, p. 531; Pai, p. 532.

Quantum Mechanics

Borneas, M.; et Kremmer, I. Au sujet de l'utilisation des polyopérateurs. Acad. R. P. Romîne. Baza Cerc. Şti. Timişoara. Stud. Cerc. Şti. Ser. I. **2** (1955), 51–58. (Romanian. Russian and French summaries)

The authors described the application of polyoperators [see Borneas, same Stud. **1** (1953), 31–34] in obtaining equations of motion of quantum-mechanical systems. He also explains that associate matrices [Gr. C. Moisil, Associated matrices of systems of partial differential equations, Editura Acad. R. P. Române, 1950; MR **16**, 1028] can be used in the computations. *A. Erdélyi.*

Araki, Gentaro. Partial description of quantum-mechanical system. Progr. Theoret. Phys. **16** (1956), 197–211.

A general discussion is given of the quantum-mechanical problem of the motion of a sub-system abstracted from the whole. When certain energy-dependent terms are small in the states considered, the effect of the rest of the system is given by potential terms of two types: one, an average background field, the other a modified inter-particle potential between the degrees of freedom in question. The theory of collective motions is of this type. Applications are given to the one-dimensional parts of π-electron interactions in molecules, particularly linear chains, where plasma oscillations due to the long-range tails of π-electron coulomb interactions account well for experimental light-asborption data. *S. Deser.*

Tobocman, W. Transition amplitudes as sums over histories. Nuovo Cimento (10) **3** (1956), 1213–1229.

It is well-known that Feynman has expressed the transition amplitudes of a quantum system as sum over "paths" or "histories" involving the action integral in an exponential. This holds for systems where the quantization is expressed by commutation relations (Bose quantization) and is proved again by the author, starting from the hamiltonian formalism. The author investigates then whether a similar result can be obtained for systems with Dirac quantization; i.e. with quantization in terms of anticommutators. Although an expression of the transition amplitude as sum over histories is obtained, it has a form very different from the Feynman integral. Use is made, for Dirac quantization, of an anticommuting field of c-numbers. A brief discussion is given of the relation of the author's results to the Lagrange formulation of Schwinger. *L. Van Hove* (Utrecht).

Halbwachs, Françis. Nouvelle représentation du fluide à spin relativiste. C. R. Acad. Sci. Paris **243** (1956), 1098–1100.

This brief note is devoted to further study of a hydrodynamical model for a relativistic particle with spin introduced by Weyssenhoff and Raabe [Acta Phys. Polon. **9** (1947), 7–18; MR **14**, 213]. *L. Van Hove.*

Destouches, J. L. Quantization in the functional theory of particles. Nuovo Cimento (10) **3** (1956), supplemento, 433–468.

Corinaldesi, E.; and Zienau, S. On an inequality for the momentum derivative of the scattering phase. Proc. Cambridge Philos. Soc. **52** (1956), 599–600.

Wigner [Phys. Rev. (2) **98** (1955), 145–147; MR **17**, 114] has derived an inequality for the derivative of the scattering phase shift to the incident particle momentum. He assumed that outside a certain radius a from the scattering centre the wave-function is that of a free particle and used his general theory of reactions. The inequality is here simply derived for scattering at any angular momentum by a spherically symmetrical potential, which vanishes beyond $r=a$. *C. Strachan* (Aberdeen).

Shimazu, Haruo. A relativistic wave equation for a particle with two mass states of spin 1 and 0. Progr. Theoret. Phys. **16** (1956), 287–298.

The author discusses the derivation of a relativistic wave equation involving nineteen component wave functions which describes a particle having two different mass states of spin 1 and zero. The energy is shown to be positive definite and the usual method of quantization is applied. *A. H. Taub* (Urbana, Ill.).

Kolsrud, Marius. Exact quantum dynamical solutions for oscillator-like systems. Phys. Rev. (2) **104** (1956), 1186–1188.

This paper contains a summary of the results of another detailed paper to be published elsewhere. It gives the exact solution of the equation $ihdT/dt=HT$ for the time-displacement operator T, if the Hamiltonian can be expressed as

$$H=\tfrac{1}{2}A_{ij}p_ip_j+a_ip_i+\tfrac{1}{2}B_{ij}q_iq_j+b_iq_i+\tfrac{1}{2}C_{ij}(p_iq_j+q_jp_i),$$

where the coefficients are arbitrary functions of time. The method is illustrated by the example of a three-dimensional oscillator in two time-dependent homogeneous fields, one of which is the conventional magnetic field with the corresponding induced circular electric field. *M. J. Moravcsik* (Upton, N.Y.).

Davies, H. Summation over Feynman histories: the free particle and the harmonic oscillator. Proc. Cambridge Philos. Soc. **53** (1957), 199–205.

The Feynman method of summation over histories of a system is illustrated on the examples of the free particle and the harmonic oscillator. The same propagators are obtained as in conventional quantum mechanics. The method is that of a particular parametrization of the path. In both problems under consideration the classical variational treatment is also given for comparison. It is claimed that the present method has the advantage of unambiguity and clarity over a different method applied to the free particle problem by Davison [Proc. Roy. Soc. London. Ser. A. **225** (1954), 252–263; MR **16**, 319]. *M. J. Moravcsik* (Upton, N.Y.).

Moses, Harry E. The Kohn-Hulthén variational procedure for the scattering operator and the reactance operator. I. Elementary form of the variational principles. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **CX-26** (1956), i+31 pp.

The author gives the variational principles as they follow from the differential or integral equation formulation of the Schroedinger equation for both the scattering matrix T and the reactance K. Both are connected with the S matrix via the equation $S=1-2\pi iT$ on the energy shell and $S=(1-i\pi K)/(1+i\pi K)$. The form following from the differential equation is known as the Kohn-Hulthen variational principle, the other the Schwinger variational principle. Four illustrations are given: (1) one-dimensional problem, (2) radial equation, angular momentum equal to zero, (3) three-dimensional scattering problem, (4) scattering of an electron by an atom. The paper concludes by restatement of the result in abstract operator form. *H. Feshbach* (Cambridge, Mass.).

Moses, Harry E. The Kohn-Hulthén variational procedure for the scattering operator and the reactance operator. II. Procedures independent of the normalization of the trial functions. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **CX-27** (1956), i+23 pp.

The Kohn-Hulthen variational procedures described in the paper reviewed above are now modified so that they will be independent of the normalization of the trial function. The essential step is the use of a trial function which consists of the unperturbed wave function plus a constant times a trial function. The constant is varied, and the resulting variational value is substituted back in the original variational integral. *H. Feshbach.*

Watanabe, Yoiti. Constraint in a quantum system. Progr. Theoret. Phys. **16** (1956), 534–536.

The author discusses the treatment of constraints in quantum-mechanical systems, and in particular describes their effect on the canonical commutation relations, and comments on their relation to the (less stringent) subsidiary conditions. *I. E. Segal* (Chicago, Ill.).

Bergmann, P. G. Introduction of "true observables" into the quantum field equations. Nuovo Cimento (10) **3** (1956), 1177–1185.

The author's previous work on the quantization of systems containing constraint equations [P. G. Bergmann and I. Goldberg, Phys. Rev. (2) **98** (1955), 531–534; MR **17**, 565] is applied to the system of an electromagnetic field which has as sources point particles subject to the Lorentz force generated by the field. The constraint equations in this case express the vanishing of the fourth component of the field momentum, π^4, and the fact that the divergence of the spacial momenta, π^s is equal to the charge density. The quantization procedure involves first the elimination from the classical Hamiltonian of the constraint momenta and their conjugate variables, here π^4, the scalar potential and the longitudinal components of π^s and the vector potential. The quantization can then be expressed in terms of Dirac brackets. The quantization procedure is shown to be Lorentz invariant by constructing the generator of an infinitesimal Lorentz transformation and showing that it is a constant of the motion.

The paper concludes with a discussion of some of the properties of bilinear combinations of Dirac field operators. *R. Arnowitt* (Syracuse, N.Y.).

Pomerančuk, I. Vanishing of the renormalized charge in electrodynamics and in meson theory. Nuovo Cimento (10) **3** (1956), 1186–1203.

This paper is the first detailed account in English of the work done by Landau and collaborators in Moscow about the high energy behaviour of quantized field theories. As such, it should be very much appreciated by all those who, like the present reviewer, are not able to read the original Russian papers. The main line of the argument is the following. The author works with the "one electron

propagation function", the "one photon propagation function" and the "vertexpart" as fundamental quantities. These three functions are related through two well-known integral equations. To get a closed system of equations that can be mathematically discussed, the author has to get an approximative third equation involving these three functions and nothing else. For this purpose the vertexfunction $\Gamma_\mu(p, p-k; k)$ is expanded in powers of the coupling constant α. Each coefficient in this expansion is a function of the three squares p^2, $(p-k)^2$ and k^2. It appears that every coefficient of this kind has a simple asymptotic form when one or more of these squares is very large. It is then argued that the asymptotic form of the vertex function itself is given as the sum of the asymptotic forms of the coefficients. This gives the missing, third equation. That such an argument is mathematically unreliable is shown by the following elementary counter-example. Consider the series

$$F(\alpha, x)=\sum_1^\infty \frac{\alpha^n}{n!}\frac{x^n}{(1+n^{-4}x^2)^n}.$$

Here, the limit of each coefficient of α^n is zero for large values of x. Nevertheless, the limit of the sum is easily shown to behave as $\exp(\alpha x)(1+O(1/x))$ for large, real values of x. The reason for this phenomenon is the fact that the coefficients are not uniformly small for large values of x. It is the reviewer's opinion that there are good physical reasons to expect the corresponding estimate in field theory to be non-uniform. In field theory, each coefficient does not reach its asymptotic form until the last threshold for the creation of real particles is passed. In $2n$ th order perturbation theory essentially n pairs can be created. Therefore, the condition for the corresponding coefficient in an expansion in powers of α to reach its asymptotic form is not $-p^2 \gg m^2$ but (at least) $-p^2 \gg 4n^2 \cdot m^2$, i.e. a definitely non uniform estimate. The present paper tries to justify the original argument by estimating the higher terms in the asymptotic expansion of the coefficients. In the example above this would mean that one estimates the coefficients not by $n^{4n}x^{-n}/n!$ but by this expression multiplied by the factor $1-n^5/x^2$. Even if it is shown that these "correction terms" are small for a fixed value of n and sufficiently large values of the "energies" p^2/m^2 etc., that is hardly enough to justify the original expansion. However, once this kind of argument is accepted, the author shows by very explicit and detailed calculations that the theory of coupled fields contains an inconsistency. This inconsistency can only be removed if it is assumed that the renormalized charge is zero, i.e. that there is no interaction. The result is also shown to be comparatively independent of the details of a special cut off procedure and of the details of the particular interaction studied.

G. Källén (Copenhagen).

Pomeranchuk, I. Ya.; Sudakov, V. V.; and Ter-Martirosyan, K. A. Vanishing of renormalized charges in field theories with point interaction. Phys. Rev. (2) **103** (1956), 784–802.

The work of Landau, Abrikosov and Khalatnikov [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 497–500, 773–776, 1177–1180; MR **16**, 315, 316] on the solutions of the quantum field theory equations is here rederived on the basis of a different limiting process involving cutoffs on both meson and nucleon momenta. These results are obtained here by letting the nucleon cutoff tend to infinity more rapidly (at various rates) than the meson one. It is claimed that the contributions of other than leading terms in the solutions now vanish independently of the size of the unrenormalized coupling constants. It is concluded that both for pseudoscalar and pseudovector coupling meson theories (as well as electrodynamics), the renormalized coupling constants, and with it the interactions, tend rigorously to zero with increasing cutoff momenta (again regardless of the size of the unrenormalized charge). High-energy electron experiments are suggested to discover any breakdown in the theory; this is expected to occur at distances of a fermi, if the cause of the inconsistency is the same for meso- and electrodynamics.

S. Deser (Copenhagen).

Gutzwiller, Martin. Quantum theory of wave fields in a curved space. Helv. Phys. Acta **29** (1956), 313–338.

This paper investigates the Klein-Gordon, Maxwell and Dirac equations as well as their quantization in a four-dimensional space-time continuum of constant curvature with space-like geodesics of finite length and time-like geodesics of infinite length. The space-time continuum is represented by a hypersurface imbedded in a five-dimensional flat space. The free wave equations for the scalar and the electromagnetic fields are studied. They can be solved by means of generalized harmonics and Legendre functions. The Dirac equation is formulated and the solutions separated into positive and negative energy ones. Propagation functions as needed for field quantization are derived and second quantization is carried out for a Dirac field and a scalar field in scalar interaction.

L. Van Hove (Utrecht).

Ogieveckiĭ, V. I. On a possible interpretation of perturbation series in the quantum field theory. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 919–922. (Russian)

The author considers first the contribution to the Lagrangian of a classical electromagnetic field arising from the polarization of the vacuum. As noted by Ioffe [same Dokl. (N.S.) **94** (1954), 437–438; MR **15**, 917], in the case of a homogeneous constant magnetic field, perturbation theory gives for this quantity a divergent series in powers of the charge. The series is here shown to be Borel summable. This result amounts to the remark that, in the integral representation given by Schwinger [Phys. Rev. (2) **82** (1951), 664–679, e.q. 3.47, MR **12**, 889] the integrand has a convergent power series in the charge although the integral itself hasn't. Second, the author considers the more general problem of applying summability methods to the perturbation series for the scattering matrix. Here the standard series of time ordered products

$$(1)\qquad S=T\Big\{\sum_{n=0}^\infty \frac{(-i)^n}{n!}g^nH^n\Big\}$$

(gH is the integral over all space time of the interaction Hamiltonian; g is the coupling constant) is rewritten as

$$(2)\quad S=\int_{-\infty}^{\infty} ds\,\exp\Big[\frac{-is}{g}\Big]\,T\Big\{\delta(s)+\frac{\varepsilon(s)}{2}\sum_{n=1}^\infty \frac{S^{n-1}}{(n-1)!n!}\Big\}H^n.$$

The series under the integral sign in (2) can converge even when the series (1) diverges like $\sum_{n=0}^\infty [\tfrac{1}{2}n]!g^n$, as it is known to do in special cases. The author argues that in any theory in which the divergence of (1) is no worse than that just mentioned, (2) can be used to give a proof of the existence of a well defined S matrix which is a distribution in the parameter, g^{-1}. {Reviewers remark: the series under the integral will then indeed converge but the

Fourier transform indicated in (2) will not exist in general since it leads typically to expressions like

$$\int_{-\infty}^{\infty} \exp\left[\frac{-is}{g}\right] \frac{e(s)}{s} [\exp k^2 s^2 - 1] ds.$$

Thus, contrary to the statement of the author, (2) is just as divergent as (1). Finally, the expansion of the exponential in (2) is used to give an expansion of S in powers of g^{-1}. *A. S. Wightman* (Princeton, N.J.).

Nelkin, Mark. Meson production by mesons in the Tamm-Dancoff approximation. Phys. Rev. (2) **104** (1956), 1150–1164.

The author calculates the cross section for the production of a single additional meson in meson-nucleon collision, and compares it with the cross section for the elastic meson-nucleon scattering. He uses the charge symmetrical pseudoscalar meson theory with the pseudoscalar coupling, and he takes into account the influence of damping on the processes under consideration. The theoretical ratio of the cross sections for production and scattering is found to be considerably less than that indicated by experiments. *S. N. Gupta.*

Ganguly, S. Nucleon-pion cascades in the atmosphere produced by α-particles. J. Tech. Bengal Engrg. Coll. **1** (1956), 113–121.

Sokolow, A. A. On the possibility of the excitation of macroscopical oscillations by the quantum fluctuations ("macroatom"). Bul. Inst. Politehn. Iaşi (N.S.) **2** (1956), 39–42. (Russian)

Bopp, F. Note on a surprising result in field-mechanics. Nuovo Cimento (10) **3** (1956), supplemento, 469–473.

It is pointed out that the existence of radiationless orbits of charged particles is a general feature of classical modified Maxwell theories, such as the one previously investigated by the author [Ann. Physik (5) **38** (194_), 345–384; Z. Naturf. **1** (1946), 53–58; MR **2**, 336; **7**, 539] if no requirement is made that the radiation damping be the same as in the Lorentz-Dirac theory.

J. Leite Lopes (Pasadena, Calif.).

Unal, Burhan; et Kahan, Theo. Théorie quantique de la dispersion et formule de Kramers-Heisenberg dans le formalisme de l'opérateur statistique. J. Phys. Radium (8) **17** (1956), 1010–1012.

Les auteurs appliquent le formalisme de l'opérateur (matrice) statistique à la théorie quantique de la dispersion dans un système atomique et moléculaire.

Résumé de l'auteur.

Enatsu, H. Relativistic quantum mechanics and mass-quantization. Nuovo Cimento (10) **3** (1956), supplemento, 526–586.

This is an extension of the author's previous work [Progr. Theoret. Phys. **11** (1954), 125–142; **12** (1954), 363–379; MR **16**, 548] which attempts to introduce new principles into quantum field theory to obtain finite mass values for elementary particles. The idea is to view a fundamental particle of mass m as compounded of two "fictitious" particles of masses m_1 and m_2. If center of mass and relative co-ordinates are introduced for the two sub-particles the field equation separates into two parts: that relating to the center of mass co-ordinates which represents the external particle (and obeys a Klein-Gordon equation) and that part relating to the relative co-ordinates representing the internal particle (subject to a wave equation with potential) which has the mass self-energy as an eigenvalue to be determined. The external particle presumably obeys the field equations used in ordinary field theory while the internal particle represents the structure of the elementary particle. The technique is applied to a pseudoscalar meson interacting with nucleons. The potential in the internal wave equation is associated with the second order vacuum polarization interaction. If the ground state of the internal system is taken to have the π meson mass, then the mass of the first excited state is of the order of the K meson mass ($\sim 1000 m_e$). *R. Arnowitt* (Syracuse, N.Y.).

Borneas, M. Sur quelques domaines d'applicabilité des polyopérateurs. Acad. R. P. Romîne. Baza Cerc. Şti. Timişoara. Stud. Cerc. Şti. Ser. I. **2** (1955), 45–50. (Romanian. Russian and French summaries)

Charon, J. Sur la signification physique de l'équation relativiste de Dirac pour l'électron. J. Phys. Radium (8) **17** (1956), 893–898.

Aymard, Alix. Les équations de l'électron magnétique déduites de la théorie des champs de tétrapodes. C. R. Acad. Sci. Paris **243** (1956), 1198–1201.

A continuation of a previous note [same C. R. **243** (1956), 885–888; MR **18**, 362] by the author on the vierbein formulation of Riemannian geometry. Properties of Dirac spinors are derived in this framework.

S. Deser (Copenhagen).

See also: Schönberg, p. 462; Hull and Julius, p. 485; Callen, p. 537; Belinfante, p. 543.

Relativity

Ingarden, R. S. On a new type of relativistically invariant linear local field equations. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 56–59. (Russian)

The question is of a new type of linear differential field equations, which are relativistically invariant, i.e. invariant with respect to the transformations of four-dimensional groups of rotations, translations and inversions of rectilinear orthogonal coordinate-systems. Moreover these transformations ought not to violate the principles of the superposition and the localisability. The author analyses the equation

$$\left(\frac{\partial^2}{\partial x_1^\mu \partial x_1^\mu} + \frac{s_{12}^2}{\lambda^4} - \kappa^2\right)\chi(x_1) = -4\pi a \delta(x_1 - x_2),$$

where (author's notation) $s_{12}^2 = -(x_1^\mu - x_2^\mu)(x_1^\mu - x_2^\mu)$, $(\mu = 1, 2, 3, 4)$, $\delta(x) = \delta(x^1)\delta(x^2)\delta(x^3)\delta(x^4)$, x_1 denotes the variable field point, x_2 the position of the source, $\chi(x_1)$ the required field function, while $\lambda > 0$ and κ are constants and $\frac{1}{2}(\kappa^2\lambda^2) \neq \pm 1, \pm 2, \cdots$.

The equations of this type: 1) ensue from the variational principle, 2) allow the separation of variables for $a = 0$ and for homogeneous boundary conditions at infinity, and 3) possess a solution for $a \neq 0$. The solution is expressed by means of Laguerre polynomials.

Compared with similar investigations of M. Gel'fand and A. M. Yaglom [Z. Eksper. Teoret. Fiz. **18** (1948), 703–733; MR **10**, 583] it should be stated that here are

treated equations with variable coefficients while the field itself is not considered independently of the existence of the sources. *T. P. Anđelić* (Belgrade).

Leruste, Philippe. Représentation possible de la théorie des probabilités dans l'espace-temps relativiste (cas d'un seul point aléatoire). C. R. Acad. Sci. Paris **243** (1956), 1021–1022.

If in a Galilean frame one introduces a probability law, then, because of the Lorentz transformation, the time in another frame becomes a random variable. This can be interpreted as follows. An elementary probability on a space-surface σ in the relativistic space-time is a scalar $f_\mu d\sigma^\mu$, where the quadrivector f_μ is the density vector of the probability current. The probability assigned to a domain Σ of σ is then the flux of f_μ through Σ. This leads to a relativistic probability formalism analogous to that of a conservative relativistic fluid. *M. Loève.*

Palacios, Julio. Review of the foundations of the theory of relativity. Rev. Acad. Ci. Madrid **50** (1956), 441–443. (Spanish)

Hoang, Pham Tan. Sur l'emploi des coordonnées isothermes pour déterminer les équations du mouvement en relativité générale. C. R. Acad. Sci. Paris **243** (1956), 1292–1295.

The author shows that contrary to the assertion of Einstein, Infeld and Hoffman, the latter's equations of motion for a point singularity in the General Theory of Relativity [Einstein and Infeld, Canad. J. Math. **1** (1949), 209–241; MR **11**, 59] are not independent of the coordinate conditions. This is proved by carrying through the EIH calculations in so-called "isothermal coordinates". The resulting equations of motion contain the same types of terms as those of EIH but with different numerical coefficients. *A. J. Coleman* (Toronto, Ont.).

Brahmachary, R. L. A generalization of Reissner-Nordström solution. I. Nuovo Cimento (10) **4** (1956), 1216–1218.

A solution of the field equations of General Relativity is obtained in which the point charge of the Reissner-Nordström solution [A. S. Eddington, Mathematical theory of relativity, 2nd ed., Cambridge, 1952, p. 185] is replaced by a sphere of constant charge density with matter density and pressure varying quadratically.
A. J. Coleman (Toronto, Ont.).

Belinfante, Frederik J. Attempts at quantization of the gravitational field. Rev. Mexicana Fis. **4** (1955), 192–206.

An expository paper on the attempts at quantizing the theories of gravitation [Belinfante and Swihart, A theory of gravitation and its quantization, Dept. of Physics, Purdue Univ., 1954; Gupta, MR **13**, 804; **14**, 447; Dirac, MR **13**, 306; Pirani, Schild and Skinner, MR **13**, 306; **14**, 418; B. S. de Witt and C. M. de Witt, MR **14**, 230; Bergmann and Goldberg, MR **17**, 565].
J. Leite Lopes (Pasadena, Calif.).

Belinfante, Frederik J. Some attempts at quantization of the gravitational field. Rev. Mexicana Fis. **4** (1955), 181–191. (Spanish)

A Spanish translation of the paper reviewed above.
J. Leite Lopes (Pasadena, Calif.).

Eisenhart, Luther P. A unified theory of general relativity of gravitation and electromagnetism. III. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 878–881.

[For parts I–II see same Proc. **42** (1956), 249–251, 646–650; MR **17**, 1016; **18**, 262]. Let $g_{\lambda\mu}$ ($F_{\lambda\mu}$) be the gravitational (electromagnetic) tensor field, $M_{\lambda\mu}$ the momentum energy tensor of $F_{\lambda\mu}$, $\Gamma_\lambda{}^\nu{}_\mu$ the Christoffel symbols of $g_{\lambda\mu}$ and ∇_μ the symbol of the corresponding covariant derivative. Put

$$\Gamma_\lambda{}^\nu{}_\mu \left\{ {\nu \atop \lambda\ \mu} \right\} + g_{\lambda\mu} v^\nu, \tag{1}$$

where v^ν is a solution of

$$\nabla_\mu v_\lambda = -v_\mu v_\lambda - M_{\mu\lambda}. \tag{2}$$

Denote by $R_{\mu\lambda}$ ($H_{\mu\lambda}$) the contracted curvature tensor of $\Gamma_\lambda{}^\nu{}_\mu$ $\left(\text{of } \left\{ {\nu \atop \lambda\mu} \right\}\right)$. Then from the integrability conditions of (2) one obtains at once

$$\mathfrak{H}_{\mu\lambda} v^\lambda = \nabla_\lambda M_\mu{}^\lambda + v^\lambda M_{\mu\lambda}. \tag{3}$$

On the other hand if one requires

$$R_{(\mu\lambda)} = 0,$$

then one obtains by straightforward computation from (1)

$$\mathfrak{H}_{\mu\lambda} v^\lambda = v^\lambda M_{\mu\lambda}, \tag{4}$$

$$\mathfrak{H}_{\mu\lambda} - \tfrac{1}{2} H h_{\mu\lambda} = M_{\mu\lambda}, \tag{5}$$

so that (3) reduces to the conservation law

$$\nabla_\lambda M_\mu{}^\lambda = 0, \tag{6}$$

and (5) are the field equations. According to (2) v_λ is a gradient. *V. Hlavatý* (Bloomington, Ind.).

Mishra, R. S. The field equations of Einstein's and Schrödinger's unified theory. Tensor (N.S.) **6** (1956), 83–89.

The equations of the Einstein (Schrödinger) unified field theory involve a general quadratic tensor $g_{\lambda\mu}$, a well defined connection $\Gamma_\lambda{}^\nu{}_\mu$, its torsion vector S_μ as well as its contracted curvature tensor $R_{\mu\lambda}$ and its first derivatives. The authors expresses these objects and the corresponding equations in terms of a connection

$$\Lambda_\lambda{}^\nu{}_\mu = \Gamma_\lambda{}^\nu{}_\mu + \delta_\lambda{}^\nu p_\mu + \delta_\mu{}^\nu g_\lambda$$

where either $g_\lambda = 0$ or $g_\lambda = p_\lambda$ and p_λ is an arbitrary vector [cf. Hlavatý, J. Rational Mech. Anal. **3** (1954), 103–146, p. 135; MR **15**, 654]. *V. Hlavatý* (Bloomington).

Husain, S. I. On unified field theory of gravitation and electromagnetism. Tensor (N.S.) **6** (1956), 132–135.

It is well known that the Einstein unified field theory applied to the case of semisymmetric connection leads only to a Riemann product space. [Cf. Hlavatý, J. Rational Mech. Anal. **2** (1953), 1–52, p. 24; **4** (1955), 653–679, p. 665; MR **14**, 505; **17**, 200.] Therefore the semisymmetric connection has to be excluded from the Einstein unified theory. The author considers just this case and shows that a convenient Lagrangian leads to Einstein's field equations. Let $h_{\lambda\mu}$ ($k_{\lambda\mu}$) be the symmetric (skew symmetric) part of a general quadratic real tensor $g_{\lambda\mu}$. Let $\mathfrak{h} \overset{\text{def}}{=} \text{Det}((h_{\lambda\mu}))$ be of rank 4. The tensor $h_{\lambda\mu}$ is considered as a metric tensor. Assume the validity of the first set of Maxwell's equations

$$k_{\lambda\mu} = 2\partial_{[\lambda} k_{\mu]} \tag{1}$$

and denote by $R_{\lambda\mu}(H_{\lambda\mu})$ the contracted curvature tensor of the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ we are looking for (of the Christoffel symbols $\left\{{\nu \atop \lambda\ \mu}\right\}$ belonging to $h_{\lambda\mu}$). Assume that the variation of

$$\int |\mathfrak{h}|^{1/2}(h^{\lambda\mu}+h^{\lambda\mu})R_{\lambda\mu}\,d\tau$$

vanishes independently with respect to $g_{\lambda\mu}$ as well as with respect to $\left\{{\nu \atop \lambda\ \mu}\right\}$. Then

$$\text{a)}\quad R_{\lambda\mu}=H_{\lambda\mu}+h_{\lambda\mu}$$

$$\text{(2) b)}\quad R_{(\lambda\mu)}=0$$

$$\text{c)}\quad \partial_{\lambda}|\mathfrak{h}|^{1/2}h^{\lambda\nu}=0$$

and

$$\text{(3)}\quad \Gamma_{\lambda}{}^{\nu}{}_{\mu}=\left\{{\nu \atop \lambda\ \mu}\right\}+\delta_{\lambda}{}^{\nu}h_{\mu}.$$

(1) and (2)c), are Maxwell's equations and the equations

$$\text{(4)}\quad R_{(\lambda\mu)}=H_{\lambda\mu}=0\quad \partial_{[\omega}R_{\mu\lambda]}=0$$

which follow at once from (2) and (1) are Einstein's field equations. *V. Hlavatý* (Bloomington).

Flint, H. T.; and Williamson, E. M. A relativistic theory of charged particles in an electromagnetic and gravitational field. Nuovo Cimento (10) **3** (1956), 551–565.

Ideas from the five-dimensional theory of Kaluza and the theory of distant parallelism are used in an attempt to formulate a theory of charged particles in which geometry would express both field and particle properties. *J. Leite Lopes* (Pasadena, Calif.).

See also: Lemmon, p. 461; Štepa, p. 535; Halbwachs, p. 539; Shimazu, p. 540; Gutzwiller, p. 541.

Astronomy

Blitzer, Leon; Weisfeld, Morris; and Wheelon, Albert D. Perturbations of a satellite's orbit due to the earth's oblateness. J. Appl. Phys. **27** (1956), 1141–1149.

The free-flight motion of a satellite around a spherical earth is investigated neglecting atmospheric deceleration. This treatment is then extended by recognizing the earth's oblateness, which is represented by an additional quadrupole field. For nearly circular orbits, one finds that perturbations of the usual planar motion may be described analytically. These are characterized by a precession of the orbit plane around the equator, that is, a regression of the nodes, amounting to as much as forty miles per revolution. The large magnitude of these effects indicates that one may exploit a satellite's motion, as measured by conventional radio techniques, to determine the earth's oblateness to a new precision. (From the author's summary.) *E. Leimanis.*

Tokmalaeva, S. S. Analytical theory of the motion of the seventh satellite of Jupiter. Trudy Inst. Teoret. Astr. **5** (1956), 160 pp. (Russian)

The problem of motion of the seventh satellite of Jupiter is reduced to that of the motion of the system: Sun, Jupiter and the satellite, using the Euler-Hill-Brown theory of lunar motion as a prototype. Prior to the present author's work this method has been used successfully for the study of motion of the eighth satellite of Jupiter, by V. F. Proskurin [Uspehi Astr. Nauk **5** (1950), 245–305; Byull. Inst. Teoret. Astr. **4** (1950), 341–354]. This paper is a more comprehensive venture in the same direction.

A considerable part of the monograph deals with the exposition of the Euler-Hill-Brown method itself (Chap. 1). Having calculated by this method the inequalities of all orders up to the third inclusive with respect to the small parameters occurring in the expansions of the coordinates of the satellite (Chap. 2), inequalities of higher orders up to the sixth inclusive are calculated by the Delaunay method adapted for a satellite instead of the Moon (Chap. 3).

In Chap. 4 is carried out the passage from the complex variables used by Hill to the polar coordinates of the satellite (longitude, latitude and radius vector) obtained by the modified Delaunay method. In addition, the arbitrary constants introduced by Brown are expressed in terms of the elliptic elements of the satellite and finally tables are given for the three coordinates of the satellite. Due to the fact that the parameters connected with the seventh satellite of Jupiter are approximately five times larger than the corresponding parameters in the Lunar Theory the expansions obtained by the Delaunay method are almost on the verge of their domain of applicability. Comparison of the two methods used shows that the Delaunay method is simpler in application but less exact than the Euler-Hill-Brown method. Using this last method the coordinates of the satellite can be calculated with any preassigned degree of exactness but with more labor involved.

In the final fifth chapter explanations are given for the calculation of the right ascension and declination of the seventh satellite of Jupiter using the tables of Chap. 4. *E. Leimanis* (Vancouver, B.C.).

McVittie, G. C.; and Rogers, M. H. On the spherically symmetric non-adiabatic motion of a gas sphere. Astr. J. **60** (1955), 374–379.

In this paper a particular field of density, velocity and pressure, which is spherically symmetrical and satisfies the equations of continuity and of momentum, is found by arbitrarily prescribing that the velocity equals the distance r from the centre of the sphere times a function of the time t only. The function of time is chosen arbitrarily so that initially the gas sphere is at rest with radius R, and then expands into vacuum, reaching the radial velocity V asymptotically as $t\to\infty$. The initial density distribution is taken arbitrarily as

$$\rho=\rho_0\left\{1-\left(\frac{r}{R}\right)^n\right\}.$$

The flow becomes adiabatic as $t\to\infty$. Before that, the total energy of the gas sphere varies. The fact that the variation must be a decrease (due to radiation) is used to obtain a minimum central temperature below which the solution is physically impossible, as a function of the index n which is related to the "mass-concentration" of the initial gas sphere. For a particular central temperature, mass and radius of this initial sphere (the values used being of the order of solar values), the total energy loss is of the order of the order of the amount of energy emitted in a supernova explosion.

{The authors produce no evidence that gas under the conditions assumed in their solution would actually

radiate energy of the amount required by their solution. The fact that their arbitrarily constructed solution requires this energy loss is of course no evidence at all, since solutions of the equations of hydrodynamics exist with the rate of energy loss per unit volume per unit time equal to any function of r and t, and the authors have merely picked one of these solutions at random.} *M. J. Lighthill.*

See also: Armellini, p. 528; Quilghini, p. 528.

Geophysics

Sarkisyan, A. S. On a generalization of the theory of Ekman. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 669–675. (Russian)

Ekman [Ark. Mat. Astr. Fys. **2** (1905), no. 11] decomposed the wind-flow-vector into two components, called the gradient and the drift components. The present author determines the level of inclination of an infinite and homogeneous ocean under the action (arbitrary) of the tangential wind pressure (the gradient component). The variation of the wind with respect to latitude is calculated after the model of Åkerblom [Nova Acta Soc. Sci. Upsal. (4) **2** (1908), no. 2]. Exact and approximate expressions are obtained for the velocity components, derived from the classical equations of Ekman (loc. cit.). {In practice, however, the author's approximations are hardly feasible from a physical point of view.}

K. Bhagwandin (Oslo).

Ruzin, M. I. The influence of turbulent friction on the motion of the particles of air. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 722–727. (Russian)

★ **Bullen, Keith Edward. Seismic wave transmission.** Handbuch der Physik. Bd. 47, pp. 75–118. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956.

A summary of mathematical and related seismological considerations pertaining to the propagation of body waves in the earth. Discusses deformation theory, initial conditions, longitudinal and shear waves, travel times, velocity distributions, amplitude theory, and information obtained on the earth's deep interior, e.g. the theory of the solid inner core. {On page 92 it is stated that the best seismological observatories aim at accuracy to 0.1 second, although local geological influences and emersion onsets frequently introduce errors well outside this uncertainty. The qualification is too weak; errors often exceed 0.1 second by a factor of 20 or more.} *A. Blake.*

★ **Ewing, William Maurice; and Press, Frank. Surface waves and guided waves.** Handbuch der Physik. Bd. 47, pp. 119–139. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956.

The authors distinguish two broad categories of seismic surface waves, according as the phase velocity is greater or less than the lowest velocity of body waves in or near the channel. Typical of the first group are the Love waves, composed of horizontally polarized shear waves multiply reflected between the earth's surface and the bottom of the crust, also the refracted sofar waves. The second group comprises the true surface (or interface) waves such as those of Rayleigh or Stoneley. The principal layers concerned in the propagation of these waves in the earth are the ocean, the crust, the mantle, and the core; in addition, there are many velocity gradients and minor layerings. These complexities lead to a considerable multiplicity of types of waves. The authors give the elements of the theory, and present many seismograms on which the particular groups of waves are well illustrated. The data are quite convincing, though there are a few cases in which the ground motion may have been considerably distorted by the seismometers. *A. Blake.*

Mhitaryan, A. M. Question of distribution of atmospheric pressure on the surface of the earth in the case of non-zonal circulation. Akad. Nauk Armyan. SSR. Dokl. **21** (1955), no. 3, 97–102. (Russian. Armenian summary)

Četaev, D. N. Computation of unsteady electromagnetic fields in inhomogeneous media. Trudy Geofiz. Inst. no. **32** (159) (1956), 3–25. (Russian)

In the electrical method of geophysical prospecting by direct current and using inertialess recording, the oscillograms picture sufficiently well how the electromagnetic field builds up during the first few seconds to justify the use of these data in the geological interpretation.

In the present paper this transient phenomenon is studied in the case of two layers of different conductivity provided that a) this difference in conductivity is small, or b) the conductivity of foundation (underlying layer) is small.

Mathematical tools used for integrating Maxwell's equations are: separation of variables and Laplace transforms. Solutions are obtained in the form of definite integrals. *E. Kogbetliantz* (New York, N.Y.).

Nikitina, V. N. On anomalies of electromagnetic fields over cylindrical inhomogeneities. Trudy Geofiz. Inst. no. **32** (159) (1956), 62–93. (Russian)

The diffraction problem of an alternating electromagnetic field above an infinite horizontal cylindrical conductor of known normal section S and of high conductivity buried in the ground (halfspace) of comparatively low conductivity is studied in its two-dimensional form.

The corresponding integral equation is discussed and solved in the case when the section S is an inclined layer of negligible thickness (half-line).

E. Kogbetliantz (New York, N.Y.).

Arnold, Kurt. Beiträge zur gravimetrischen Geodäsie. Deutsche Akad. Wiss. Berlin. Veröff. Geodät. Inst. Potsdam no. **11** (1956), 30 pp.

Three contributions formally unrelated to each other are contained in the publication. The first deals with the determination of the geoidal height, N, from a series expansion of the Stokes integral formula for N. Let φ, λ be geographic coordinates. Then,

$$(*) \qquad N=\sum_{i=0}^{2}\sum_{j=0}^{2}(N_{ij})_0\Delta\varphi^i\Delta\lambda^j \quad (i+j\leq 2),$$

where (φ_0, λ_0) is a given origin, and $\Delta\varphi=\varphi-\varphi_0$, $\Delta\lambda=\lambda-\lambda_0$. Formulas for $(N_{ij})_0$ are obtained in integral form which are more complicated than the original Stokes formula. The advantage of the author's presentation, however, is twofold: first, N_{ij} are computed only once — at (φ_0, λ_0) — for all N in the neighborhood of (φ_0, λ_0); and second, their evaluation depends mainly on the gravity anomalies in the immediate neighborhood of (φ_0, λ_0). This last property becomes more pronounced,

the higher the order of N_{ij}; hence it becomes reasonable not to carry out the expansion of (*) beyond the second order (although no precise analysis of the remainder is made). The section concludes with the transformation of (*) into terms of Gauss-Kruger (transverse Mercator) coordinates, and the computation of the derivatives of ξ and η, the gravimetric deflections of the vertical. The latter formulas are compared with similar ones previously developed by de Graaff-Hunter [Proc. Soc. London. Ser. A. **206** (1951), 1–17, p. 5; MR **13**, 78] and Tengström [Rikets Allmänna Kartverk, Meddelande no. 22, Stockholm, 1954, p. 51], and differences explained.

In the second contribution, a formula for the gravimetric correction, $\Delta\bar{N}$, in astronomic levelling is derived. It is first shown that for distances, S, less than 200 km between adjacent points, only first order terms in S are required. $\Delta\bar{N}$ is conveniently evaluated by an integral approximation derived from Stokes formula, using a coordinate system transformed from the conventional polar form to a family of orthogonal circles.

In the third section, the curvature of the plumb between points P and Q is obtained by an approximation formula involving observed values of gravity, the topographic corrections at P and Q, and the change in the free air reduction between P and Q. It is shown that in mountainous terrain the last factor can be quite appreciable (of the order of 1 second of arc or more).

B. Chovitz (Washington, D.C.).

See also: Welander, p. 528; Blitzer, Weisfeld and Wheelon, p. 544.

OTHER APPLICATIONS

Games, Economics

Guy, Richard K.; and Smith, Cedric A. B. The G-values of various games. Proc. Cambridge Philos. Soc. **52** (1956), 514–526.

P. M. Grundy [Eureka 2 (1939), 6–8] has considered a disjunctive combination of a finite set of 2-person games in which the players move alternately, each in turn making a move in one and only one of the individual games. It was further assumed that (i) the player unable to move loses, (ii) the possible moves from any position depends only on the position, and (iii) the games are of finite play. Grundy has shown the existence of an integral valued function $G(P)$ of the position P with the properties that (a) $G(P)=0$ for the terminal position and for any other position $G(P)$ is the smallest non-negative integer different from all values of $G(Q_i)$ where it is possible to move from P to Q_i, (b) for a disjunctive combination of games, G is the nim-sum of the G_i's for the individual games (each $G(P)=\sum a_r 2^r$ and the a_r are added mod 2 for each value of r, i.e. binary addition without carry), and (c) the player who consistently moves to a position for which $G=0$ wins the game.

The authors consider a large collection of nim-type games satisfying these conditions. A position consists of a number of counters in rows, as in nim. The games vary in the rules governing the removal of counters from a row or the division of a row into rows, or a combination of these moves. Methods for computing G explicitly are outlined and the results for some games are reported in detail. A number of interesting properties which may be possessed by the function G are discussed. Perhaps the most interesting is the periodicity of G as a function of the number r of counters in a row for large r. The periodicity is determined for some particular games but the question is unresolved in general.

E. D. Nering (Tucson, Ariz.).

Grundy, P. M.; and Smith, C. A. B. Disjunctive games with the last player losing. Proc. Cambridge Philos. Soc. **52** (1956), 527–533.

The authors consider a class of games similar to that considered by Guy and Smith [see the preceding review] and Grundy except that the player forced to make the last move loses. Although the two situations seem to be superficially similar the class of games considered here presents considerably more difficulty. Nim may be played with the last player winning or the last player losing and the two situations are quite similar. In fact, the play is identical until the last few moves are reached. The difficulty is to discuss the situation in generality. In both cases functions of the positions are established in terms of which it is possible to describe winning strategies. The functions established here are rather more difficult to compute and use than are those established for games with the last player winning. The authors attack the problem of whether more efficient functions can be found. They show that even a "best" function is likely to be quite difficult to handle. *E. D. Nering* (Tucson, Ariz.).

Nataf, André. Possibilité d'agrégation dans le cadre de la théorie des choix. Metroecon. **5** (1953), 22–30.

The author discusses rather informally the question in the field of statistical economics as to when one can treat the sum of different goods consumed and the sum of satisfaction received by the theory applicable to a single individual making purchases among different consumer goods so as to obtain, subject to linear budget restrictions, the maximum satisfaction. By appeal to elementary kinematic notions, he concludes that a necessary and sufficient condition for such statistical grouping to be valid is that the Engel curves for the individuals be parallel straight lines. Qualitative considerations lead him to conclude that one obtains what may be properly called the indifference surfaces of the society. In the case of more than two goods, the indifference surfaces for which the Engel curves are straight lines or line segments need be discussed only in the domains of plausible variation. An addendum acknowledges relationships to conclusions obtained independently by W. M. Gorman [Econometrica **21** (1953), 63–80; MR **14**, 1000]. *A. A. Bennett.*

Kuhn, H. W. A note on "The law of supply and demand." Math. Scand. **4** (1956), 143–146.

A new and shorter proof is given of the following lemma due to the reviewer [Math. Scand. **3** (1955), 155–169; MR **17**, 985]. Lemma. Let S be a bounded continuous set valued function from the unit $(n-1)$-simplex into R_n such that (a) $S(p)$ is non-empty and convex for all $p \in P$, and (b) if $x \in S(p)$, then $x \cdot p \geqq 0$. Then there exists $p \in P$ and $x \in S(p)$ such that $x \geqq 0$. The new proof is based on a fixed-point theorem of Eilenberg and Montgomery [Amer. J. Math. **68** (1946), 214–222; MR **8**, 51]. It is also shown that the minimax theorem of game theory in a simple consequence of the above lemma. *D. Gale.*

Roy, René. La demande des biens indirects. Metroecon. 4 (1952), 66–78.

Taking the chain corn — flour — bread as typical simple case the author elucidates the complex relationships between direct and indirect demand. *H. Wold.*

Wagner, Harvey M. An eclectic approach to the pure theory of consumer behavior. Econometrica 24 (1956), 451–466.

Although all regularity assumptions are abandoned, a weak axiom of preference transitivity suffices to establish certain basic propositions, e.g. that if prices are constant the consumer's expenditure is a nondecreasing function of income. *H. Wold* (Uppsala).

Palomba, Giuseppe. La teoria matematica del bilancio contabile. Ricerca, Napoli 7 (1956), 3–17.

Gordon, Myron J.; and Taylor, William J. The condition for lot size production. J. Amer. Statist. Assoc. 51 (1956), 627–636.

Banerjee, K. S. Simplification of the derivation of Wald's formula for the cost of living index. Econometrica 24 (1956), 296–298.

The author considers Wald's well-known index formula [Econometrica 7 (1939), 319–331; MR 1, 24]

$$P_{01}=\frac{\sum p_1(q_0+[\mu/\lambda]^{\frac{1}{2}}q_1)}{\sum p_0(q_0+[\mu/\lambda]^{\frac{1}{2}}q_1)},$$

where μ and λ denotes quantities depending on the marginal utility of money. The purpose of the author is to simplify Wald's derivation. The basic assumptions are the same, i.e., the utility indicator function $I(q)$ can be approximated by a polynomial of the second degree and the Engel curves are linear within the narrow range considered. *L. Törnqvist* (Helsinki).

Downton, F. On limiting distributions arising in bulk service queues. J. Roy. Statist. Soc. Ser. B. 18 (1956), 265–274.

The (equilibrium) distributions of queue size, obtained by Bailey [same J. 16 (1954), 80–87; MR 16, 148], and of waiting time, obtained by the author [ibid. 17 (1955), 256–261; MR 17, 867], in queuing systems with bulk service, are examined for their limits with increasing batch size, supposing the service time distribution of χ^2 type. For fixed batch size, the distributions are both expressed in terms of the roots of a certain equation; the same is true for the limiting distributions, the limiting equation becoming transcendental and curiously the same as that found by Lindley [Proc. Cambridge Philos. Soc. 48 (1952), 277–289; MR 13, 759] for a single server system with regular arrivals and χ^2 service time (for reasons explained by the author). A special examination is given for constant service time, which is a limiting case. Tables are given for the average and variance of both waiting time and queue size with varying batch size and service time distribution, including both limits.

J. Riordan (New York, N.Y.).

Thomissen, F. X. The frequency of industrial accidents. Statistica, Neerlandica 10 (1956), 163–176. (Dutch summary)

★ **Beale, E. M. L. Linear programming by the method of leading variables.** Conference on linear programming, May, 1954, pp. 56–71; discussion, 72. Ferranti Ltd., London.

This paper contains a summary of the method of leading variables developed by the author in an earlier paper [Proc. Cambridge Philos. Soc. 50 (1954), 512–523; MR 16, 155]. A formal computing procedure is outlined and two examples are worked out in detail. *E. D. Nering.*

★ **Land, A. A problem in transportation.** Conference on linear programming, May, 1954, pp. 20–28; discussion, 30–31. Ferranti Ltd., London.

Mrs. Land presents here a preliminary report on her study of the problem of transporting coal from the collieries to the coke ovens, to see if application of the methods of linear programming — in this case, of course, the transportation problem — could result in savings for the National Coal Board of Great Britain. There is an interesting discussion of associated economic questions. *A. J. Hoffman* (New York, N.Y.).

Eisemann, Kurt. Simplified treatment of degeneracy in transportation problems. Quart. Appl. Math. 14 (1957), 399–403.

Several methods have been proposed in order to reduce the number of iterations to be performed before reaching the final solution of the transportation problems. The common feature to these methods is the computation of the initial solution according to the relative size of the entries in the cost matrix. [See Houthakker, J. Operations Res. Soc. Amer. 3 (1955), 210–214; MR 16, 962; de Lasala, Trabajos Estadist. 6 (1955), 209–236; MR 17, 1106; and other references given by the author.] However, these procedures lead, in some cases, to defective initial solutions with a number q of non-zero values smaller than $m+n-1$ if the matrix of the problem is of size $m\times n$. Then the problem arises of selecting $m+n-1-q$ new variables of zero value to be included as members of the initial solution, obtaining a "basis" in such a way that degeneracy can be avoided throughout the whole process. The author presents a very simple device for computing directly the non-negative $m+n-1$ values corresponding to the initial basis, which excludes the possibility of difficulties connected with degeneracy arising in the following stages. The amount of work involved is negligible and the method can be used for hand computations as well as for electronic machine computing. *A. G. Azpeitia.*

Charnes, A.; and Miller, M. H. A model for the optimal programming of railway freight train movements. Management Sci. 3 (1956), 74–92.

The authors state in their introduction that "the area of railroad scheduling and associated problems offers a potentially fruitful field for research in management science", and more than justify this opinion in the text which follows. The central point of interest mathematically is the discussion of the linear programming problem in which the matrix A has the following form: let N be a positive integer, let the rows of A refer to all possible pairs (i, j), $i\neq j$, $i, j=1, \cdots, N$, and let the columns of A refer to all possible cycles of two or more numbers from 1 to M. Then A contains a 1 in the entry corresponding to row (i, j) in any column containing a cycle in which j follows i, and 0 in all other columns. The authors develop an efficient algorithm for solving this linear programming

problem and they believe that the matrix A may be sufficiently important to merit inclusion among the basic model types of linear programming. In the particular case under study, the lengths of the cycles were no greater than 3, though the method is applicable in general. {The reviewer regrets, however, that the situation in which the solution may turn out to be fractional (which can happen, even in their case, if $N \geqq 6$) was not interpreted or even discussed beyond the hint that it might arise.}

A. J. Hoffman (New York, N.Y.).

Bellman, Richard. Dynamic programming and Lagrange multipliers. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 767–769.

The purpose of this note is to indicate how a suitable combination of the classical method of the Lagrange multiplier and the functional-equation method of the theory of dynamic programming [cf. Bellman, Bull. Amer. Math. Soc. **60** (1954), 503–515; MR **16**, 732] can be used to solve numerically, and treat analytically, a variety of variational problems that cannot readily be treated by either method alone.

From the introduction.

See also: Heller and Tompkins, p. 459; Beale, p. 514; Wagner, p. 515; Dantzig and Fulkerson, p. 536.

Biology and Sociology

Rashevsky, N. What type of empirically verifiable predictions can topological biology make? Bull. Math. Biophys. **18** (1956), 173–188.

In einer vorangehenden Serie von Arbeiten über topologische Biologie [Bull. Math. Biophys. **16** (1954), 317–348; **17** (1955), 45–50, 111–126, 207–218, 229–235; **18** (1956), 31–56, 113–128; MR **16**, 386, 663, 1044; **17**, 280, 761, 1106] wurde vom Verfasser darauf verwiesen, dass das metrische Verfahren in der mathematischen Biologie, das ja ohne Zweifel praktische Erfolge gehabt hat, seine Grenzen haben muss, weil viele biologische Gesetzmässigkeiten nicht durch quantitative Gesetze sondern durch einfache Zusammenhänge beschrieben werden. Zur Darstellung dieser Zusammenhänge — die jedenfalls biologisch ebenso wichtig wie die quantitativen Gesetze sind — ist eben die Topologie die geeigneteste Methode.

Bezüglich der topologischen Struktur des Urorganismus kann man verschiedene Hypothesen machen. Es werden dabei die 10 vegetativen und die 7 animalen Funktionen eines Lebewesens berücksichtigt. Diese „Urgraphen" werden dann transformiert, wobei der Verfasser zuerst 14 solche verschiedene Transformationen berücksichtigt. In der vorliegenden Arbeit wird gezeigt, dass die spezifischen Voraussagen zwar von den eingeführten spezifischen Annahmen abhängen, der Typ dieser Zusammenhänge jedoch charakteristisch für das topologische Verfahren ist.

Als Beispiel wird die Zahl der möglichen Organismen berechnet, das Problem des Zusammenhanges zwischen der Zahl der Organe und der der Zelltypen, die Frage der Zahl der Zelltypen in den verschiedenen Organen und das Problem der endokrinen Sekretionen werden besprochen. Es sei hier nur noch erwähnt, dass bezüglich der angewandten mathematischen Methode der Verfasser sich auf das bekannte Werk D. König, Theorie der endlichen und unendlichen Graphen [Akademische Verlagsgesellschaft, Leipzig, 1936] beruft. *Th. Neugebauer.*

Hetz, Wolfgang. Untersuchungen zum Erneuerungsproblem, insbesondere seine Darstellung als Markoffsche Kette. Bl. Deutsch. Ges. Versicherungsmath. **3** (1956), 23–56.

The renewal problem of population theory is treated as a Markov chain. Results that were only implicit in, and scattered through, the previous literature [M. S. Bartlett, Stochastic processes, Univ. of North Carolina, 1947; E. G. Lewis, Sankhyā **6** (1942), 93–96; P. H. Leslie, Biometrika **33** (1945), 183–212; **35** (1948), 213–245 [MR **7**, 465; **10**, 386]; none of which are cited by the author] are now explicitly obtained. These include the spectral resolution of the matrix of transition probabilities — which is extended to the case where replacements occur in fixed proportions at a number of ages (instead of at age 0), the proof of the convergence of these probabilities, and the derivation of the distribution of the renewals at time t ($t=0, 1, 2, \cdots$). A novelty is the application of a theorem of Hoeffding and Robbins [Duke Math. J. **15** (1948), 773–780; MR **10**, 200] to obtain the asymptotic distribution of the replacements occurring up to time t. Finally the author formally generalizes the theory to include the case where each individual of type A_j ($j=1, 2$) may be replaced by an individual of type A_k ($k=1, 2$) with probability α_{jk} ($\alpha_{j1}+\alpha_{j2}=1$). *H. L. Seal* (New York, N.Y.).

Rosner, Burton S. A new scaling technique for absolute judgments. Psychometrika **21** (1956), 377–381.

The results of an experiment using the method of absolute judgments can be viewed as a matrix of conditional probabilities in which the rows represent stimuli and the columns responses. The cosine of the angle between two row vectors is a measure of the similarity of the corresponding stimuli. This cosine provides the basis for a method of scaling the stimuli. Unlike the method of paired comparisons, this new technique does not require arbitrary fixing of a unit of measurement. A numerical example is given. *Author's summary.*

McHugh, Richard B. Efficient estimation and local identification in latent class analysis. Psychometrika **21** (1956), 331–347.

Estimators which are efficient in the sense of having minimum asymptotic variance are obtained for the structural parameters of Lazarsfeld's latent class model of latent structure analysis. Sufficient conditions for the local identification of the structural parameters are also presented. *Author's summary.*

Bock, R. Darrell. The selection of judges for preference testing. Psychometrika **21** (1956), 349–366.

A scheme for choosing a few individuals whose preferences for given objects are most representative of those of a larger group of individuals is proposed. The method involves (a) quantifying the preferences of each individual so as to discriminate optimally among objects, (b) testing statistically whether or not a common preference continuum may be assumed for the quantified preferences, (c) constructing a linear estimator of values for the objects on this continuum, if it may be assumed, and (d) selecting as judges the least number of individuals whose quantified preferences, when used with this estimator, determine values for the objects with acceptable accuracy. A numerical example based on food preferences is presented. *Author's summary.*

See also: Kokoszýnska, Kubiński and Słupecki, p. 457; Weber, p. 521; Rapoport, p. 549.

Information and Communication Theory

Rapoport, Anatol. On the application of the information concept to learning theory. Bull. Math. Biophys. **18** (1956), 317–321.

A learning curve derived by H. D. Landahl [same Bull. 3 (1941), 71–77] from postulated neurological structures is shown to be derivable from simplified assumptions by introducing the information measure of the uncertainty of response. The possible significance of this approach to learning theory is discussed. (Author's abstract)

A. S. Householder (Oak Ridge, Tenn.).

Control Systems

★**Hahn, Wolfgang. Nichtlineare Regelungsvorgänge.** Verlag R. Oldenbourg, München, 1956. 108 pp. DM 11.20.

This book of the five lectures, by different authors, which were given at a conference held in Darmstadt in 1955. The lectures deal with methods for the theoretical study of nonlinear control systems, and with various properties of these systems. Specifically, one lecture gives a very readable introduction to the stability theory of Liapounoff, one deals with an on-off control system having hysteresis, and the other three deal with the method, or group of methods, commonly designated by the terms "describing function method", "method of Kryloff and Bogoliuboff", and "principle of harmonic balance". Although no very novel ideas are presented, the examples discussed and the treatment of particular details give the book many points of interest. In a few places excessive conciseness, coupled with a rather unfamiliar engineering terminology, makes for difficult reading.

L. A. MacColl (New York, N.Y.).

Kuhtenko, V. I. On simultaneous reproduction of accidental and non-accidental signals by linear tracking systems. Avtomat. i Telemeh. **17** (1956), 425–430. (Russian)

An extension of Wiener's theory to the case where the signal contains a known non-random component. The optimal system is defined as one which (i) passes the non-random component with no distortion, and (ii) minimizes the mean-square value of the random component of the error. It is shown that the impulsive response, $k(t)$, of an optimal system satisfies a Wiener-Hopf equation whose right-hand member involves terms of the form

$$\int_0^\infty dt\, k(t) \cos \beta_\lambda t \exp(-\Delta+\alpha_\lambda)t,$$

where the $p_\lambda=\alpha_\lambda+i\beta_\lambda$ are poles of the non-random component and Δ is a constant. A general solution of this equation is found. *L. A. Zadeh* (New York, N.Y.).

Popov, E. P. Approximate investigation of transition processes in non-linear automatic systems by the method of harmonic linearization. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1956, no. 9, 3–23. (Russian)

An extension of the method of harmonic linearization to the case of rapidly decaying oscillations. Several examples are treated in detail. *L. A. Zadeh.*

Moore, E. F.; and Shannon, C. E. Reliable circuits using less reliable relays. I, II. J. Franklin Inst. **262** (1956), 191–208, 281–297.

The problems considered here are closely akin to the problems of control of errors in automata studied by von Neumann [Automata studies, Princeton, 1956, pp. 43–98; MR 17, 1040]; here the automata being relay machines. Given an unreliable relay, let a be the probability of a contact being closed when its coil is energized and c the probability of a contact being closed when its coil is not energized. The authors then show that so long as the contacts are statistically independent and that $a \neq c$, it is possible to construct, out of these relays, networks with arbitrarily good reliability. This is accomplished by an iterative process in which each individual contact of a network is replaced by certain network of contacts. The authors go beyond showing existence by presenting one specific method of construction and give, for known values of a, c and desired reliability, the number of contacts sufficient to achieve this reliability. The number of contacts necessary to attain a given reliability is also given for this model and the authors show that these two numbers are quite close to each other for several cases considered.

It is rather remarkable to note that whereas von Neumann needed a redundancy of something like 60,000 to achieve a certain reliability in one example, the relay model needs a redundancy of only about 100 to achieve the same reliability. Basically, a relay designer has at his disposal certain geometrical properties of relay networks which are inaccessible to, say, a diode logic designer. This and the fact, as remarked in the paper, that logical combinations of relay circuits are assumed in the model to be error-free perhaps account for the difference in the two cases. *C. Y. Lee* (Chatham, N.J.).

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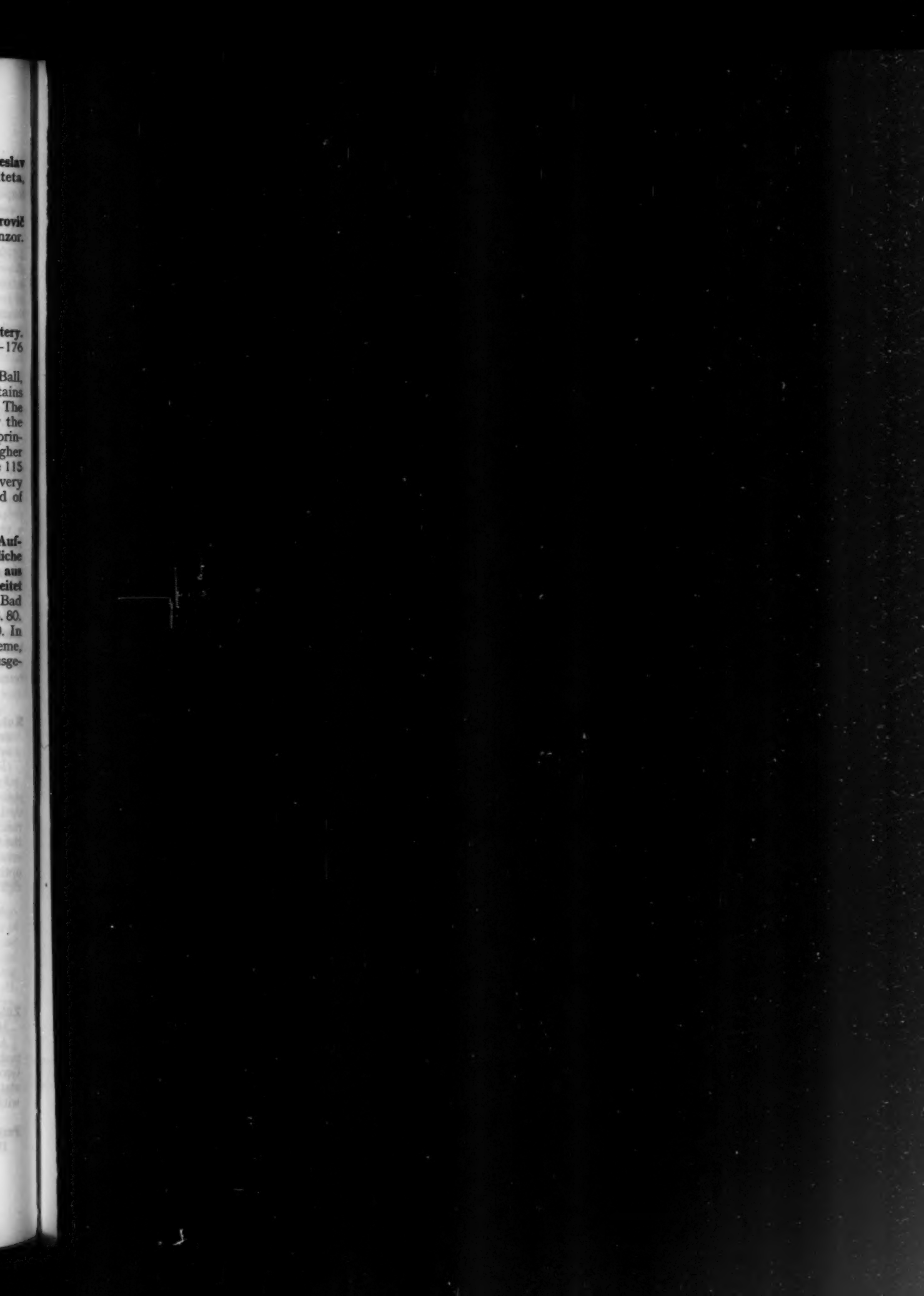
"The classic reference work of W. W. Rouse Ball, namely Mathematical Recreations and Essays, contains many early examples of mathematical conjuring ···. The present book represents the first attempt to survey the entire field of modern mathematical magic···. Its principles can be grasped quickly, without training in higher mathematics." (From the author's preface.) There are 115 tricks depending on cards, topology etc. In almost every trick, not only the final effect but also the method of operation is entirely mathematical.

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